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Universal higher order singlet QED corrections to unpolarized lepton scattering

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Abstract. We calculate the universal flavor singlet radiative QED corrections to unpolarized lepton scattering applicable to general differential scattering cross sections, involving charged fermions or photons in the initial or final states. The radiators are derived to $O((\alpha \ln(Q^2/m_f^2))^5)$ in analytic form. Numerical illustrations are given.

1 Introduction

QED corrections to integral and differential cross sections of light charged lepton–anti-lepton scattering or deeply inelastic lepton–nucleon scattering turn out to be quite large in some kinematic regions [1–13]. This applies in particular to the Bremsstrahlung contributions due to significant shifts in the kinematics of the underlying differential scattering cross sections. The universal corrections can be grouped into flavor non-singlet and flavor singlet contributions. In the orders $O((e_f^2 \alpha L)^k)$, with α the fine structure constant, $L = \ln(Q^2/m_f^2)$, Q^2 the typical virtuality of the process, and m_f, e_f the fermion mass and charge, respectively, the non-singlet contributions stem from the leading order anomalous dimension in QED, $P_{ff}(x, Q^2)$. The diagonal elements of the singlet anomalous dimension matrix contain a $\delta(1-x)$ distribution and are distribution-valued due to $(\dots)_+$ distributions. Therefore the numerical Mellin inversion does badly converge in the region of $x \lessapprox 1$. Analytic representations are required to high enough order k in the fine structure constant α to cover all universal effects for the energy ranges probed at present day colliders and those to be built in the foreseeable future. This applies to high luminosity experiments at future linear colliders [14, 15] and as well to the search for rare reactions at LHC. The second order universal corrections for various processes have been known for a long time [16–20]. The third order corrections were given in [21][1] for the flavor non-singlet and in [23] also for the singlet case. Later the fifth order non-singlet corrections were given in [24] and recalculated in [25] and [26–29].[2] In [26] a very compact form was given for the non-singlet contributions, which are the same for polarized and unpolarized scattering. There also the polarized singlet contributions were calculated to $O((\alpha L)^5)$. In the present paper the unpolarized singlet evolution kernels are calculated to $O((\alpha L)^5)$, which supplements earlier investigations for the flavor non-singlet kernel [26]. A second class of universal QED corrections was treated previously in the non-singlet [30, 31] and polarized singlet case [26]. It concerns the leading order small-x resummations of $O((\alpha \ln^2(x))^k)$. These resummations are based on corresponding resummations in QCD [32–37]. In the unpolarized (pure) singlet case the leading order small-x QCD resummations [38] result from the non-abelian gluon coupling, which is absent in QED. Therefore a transformation of the respective kernels is not possible in this case.

The paper is organized as follows. The general framework to derive the radiators in $O((\alpha L)^k)$ is outlined in Sect. 2. In Sect. 3 the leading order singlet radiators are calculated to $O((\alpha L)^5)$. In Sect. 4 numerical illustrations are given, and Sect. 5 contains the conclusions. An appendix lists useful convolution relations, which were needed in the calculation and are of use in other QED and QCD calculations.

2 The solution of singlet evolution equations

The universal QED corrections $O((\alpha L)^k)$ for general values of the collinear radiation momentum fraction x can be expressed solving the singlet evolution equations starting at a low scale Q_0^2. This scale may be identified with a typical charged lepton mass m_f squared. The running coupling constant $a(Q^2) = \alpha(Q^2)/(4\pi)$ obeys the evolution equation

$$\frac{\mathrm{d}a(Q^2)}{\mathrm{d}\ln(Q^2)} = -\sum_{k=0}^{\infty} \beta_k a^{k+2}(Q^2)\,, \qquad (1)$$

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[1] For an application to the Z peak see [22].

[2] The results of [24] and [26] agree, but they partly disagree in the fifth order with [25].

where β_k denote the expansion coefficients of the QED β-function in the $\overline{\text{MS}}$-scheme, $\beta_0 = -4/3 N_f$, $\beta_1 = -4N_f$, $\beta_2 = 2N_f + (44/9)N_f^2$ etc. [39, 40] for N_f active charged lepton species. At leading order the solution

$$a(Q^2) = \frac{a(m_f^2)}{1 - \frac{4}{3}N_f a(m_f^2)L}\,, \quad (2)$$

with $L = \ln(Q^2/m_f^2)$ is obtained.

The universal radiators are found as solutions of the leading order QED renormalization group equations associated to the collinear singularities. The following QED distributions are of relevance:

$$D^f_{\rm NS}(a,x) = D^f(a,x) - D^{\bar{f}}(a,x)\,, \quad (3)$$
$$D^f_{\Sigma}(a,x) = D^f(a,x) + D^{\bar{f}}(a,x)\,, \quad (4)$$
$$D_{\gamma\gamma}(a,x) = D_{22}(a,x)\,, \quad (5)$$
$$D_{\gamma f}(a,x) = D_{21}(a,x)\,, \quad (6)$$
$$D_{f\gamma}(a,x) = D_{12}(a,x)\,. \quad (7)$$

The non-singlet distribution $D^f_{\rm NS}(a,x)$ was dealt with in [24–26] before. Here the $D_{ij}(a,x)$ denote the respective matrix elements of the singlet radiator $\mathbf{D}_{\rm S}(a,x)$ given below.

The singlet radiator functions at the scale Q_0^2 are

$$\mathbf{D}(Q_0^2)(x) \equiv \mathbf{D}(a_0) = \mathbf{1}\delta(1-x) \quad (8)$$

since both the charged leptons and the photon are considered to be asymptotically stable particles. The evolution equations read

$$\begin{aligned}\frac{\partial \mathbf{D}_{\rm S}(a,x)}{\partial a} &= -\frac{1}{a}\frac{\sum_{k=0}^{\infty} a^k \mathbf{P}_k(x)}{\sum_{k=0}^{\infty} a^k \beta_k} \otimes \mathbf{D}_{\rm S}(a,x) \\ &= -\frac{1}{\beta_0 a}\left[\mathbf{P}_0(x) + a\left(\mathbf{P}_1(x) - \frac{\beta_1}{\beta_0}\mathbf{P}_0(x)\right)\right. \\ &\qquad \left. + O(a^2)\right] \otimes \mathbf{D}_{\rm S}(a,x)\,. \end{aligned} \quad (9)$$

The Mellin convolution is defined by

$$A(x) \otimes B(x) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \delta(x - x_1 x_2) A(x_1) B(x_2)\,. \quad (10)$$

The Mellin transform

$$\mathbf{M}[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x) \quad (11)$$

diagonalizes the convolution (10) to

$$\mathbf{M}[A(x) \otimes B(x)](N) = \mathbf{M}[A(x)](N) \cdot \mathbf{M}[B(x)](N)\,. \quad (12)$$

In some of the radiators $(\ldots)_+$ distributions emerge, which are defined relative to the set of smooth test functions $\phi(x)$ with compact support by

$$\int_0^1 \mathrm{d}x [F(x)]_+ \phi(x) = \int_0^1 \mathrm{d}x F(x)[\phi(x) - \phi(1)]\,. \quad (13)$$

Equation (9) may be solved easiest in Mellin space as a matrix-valued ordinary differential equation to all orders; see e.g. [41, 42]. The leading order solution reads

$$\begin{aligned}\mathbf{D}_{\rm S,0}(a,x) &= [\exp(-\mathbf{R}_0(x)\ln(a/a_0))_{\otimes}] \otimes \mathbf{D}_{\rm S}(a_0,x) \\ &\equiv \mathbf{E}_0(a,a_0,x) \otimes \mathbf{D}_{\rm S}(a_0,x)\,, \end{aligned} \quad (14)$$

where $\mathbf{R}_0 = \mathbf{P}_0/\beta_0$, and $\mathbf{E}_0$ denotes the leading order singlet evolution operator. We use the short-hand notation

$$[f(g(x))_{\otimes}] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \otimes_{l=1}^{k} g(x)\,, \quad (15)$$

with $\otimes_{l=1}^{k}$ the k-fold convolution. The singlet solution (14) to kth order in $\alpha(Q^2)L$ therefore requires one to calculate k-fold convolutions of the leading order matrix of splitting functions.

The method described above can be extended to subleading logarithmic contributions, i.e. terms of $O(\alpha^2 L)$ etc. These contributions contain process-dependent parts being described by Wilson coefficients in inclusive situations or the corresponding semi-inclusive quantities. These corrections are neither universal nor independent of the measurement chosen for the kinematic variables. Examples are the $O(\alpha^2 L)$ corrections for the initial-state radiation in e^+e^- annihilation [16, 17], the $O(\alpha^2 L)$ initial- and final-state radiation corrections to deeply inelastic scattering [43] and the $O(\alpha^2 \ln(m_\mu/m_e))$ corrections to the electron spectrum in muon decay [44].

3 The leading order solution to $O[(\alpha L)^5]$

In the following we derive the solution for the unpolarized singlet QED evolution kernels up to $O((\alpha L)^5)$. The singlet evolution equation is solved in the running coupling $a(Q^2)$. However, one may re-parameterize the representation and express the evolution kernel directly in terms of $a_0 = a(m_f^2)$ by

$$\begin{aligned}\mathbf{E}_0(a,x) &= \mathbf{1}\delta(1-x) + \mathbf{P}_0(x)a_0 L \\ &+ \left(\frac{1}{2}\mathbf{P}_0^{(1)}(x) + \frac{2}{3}\mathbf{P}_0(x)\right)(a_0 L)^2 \\ &+ \left(\frac{1}{6}\mathbf{P}_0^{(2)}(x) + \frac{2}{3}\mathbf{P}_0^{(1)}(x) + \frac{16}{27}\mathbf{P}_0(x)\right)(a_0 L)^3 \\ &+ \left(\frac{1}{24}\mathbf{P}_0^{(3)}(x) + \frac{1}{3}\mathbf{P}_0^{(2)}(x) \right. \\ &\qquad \left. + \frac{22}{27}\mathbf{P}_0^{(1)}(x) + \frac{16}{27}\mathbf{P}_0(x)\right)(a_0 L)^4 \\ &+ \left(\frac{1}{120}\mathbf{P}_0^{(4)}(x) + \frac{1}{9}\mathbf{P}_0^{(3)}(x) + \frac{14}{27}\mathbf{P}_0^{(2)}(x) \right. \\ &\qquad \left. + \frac{80}{81}\mathbf{P}_0^{(1)}(x) + \frac{256}{405}\mathbf{P}_0(x)\right)(a_0 L)^5\,. \end{aligned} \quad (16)$$

Here the matrices $\mathbf{P}_0^{(k)}$ are

$$\mathbf{P}_0^{(k)}(x) = \otimes^{(k)} \mathbf{P}_0(x)\,, \quad (17)$$

i.e. $\otimes^{(1)}\mathbf{P}_0(x) = [\mathbf{P}_0 \otimes \mathbf{P}_0](x)$ etc. The corresponding expressions for the $(1,1)$-components of $\mathbf{P}_0^{(k)}(x)$ are given relative to the non-singlet components $\mathbf{P}_{\rm NS}^{(k)}(x)$ [26]. The singlet matrices are

$$\mathbf{P}_0^{(k)}(x) = \begin{pmatrix} P_{11}^{(k+1)}(x) & P_{12}^{(k+1)}(x) \\ P_{21}^{(k+1)}(x) & P_{22}^{(k+1)}(x) \end{pmatrix}, \tag{18}$$

with components $P_{ij}^{(k)}(x)$ given below in (19)–(39). They were calculated using the convolution formulae of Appendix A and relations given in [26, 45] before. The projections $P_{ij}^{(k)}$ describe the splitting of a fermion into a fermion (1,1), of a photon into a fermion, positron, respectively (1,2), a fermion into a photon (2,1), and a photon into a photon (2,2) in kth order in the renormalized coupling constant.

The leading order QED splitting functions can be obtained identifying $T_R = C_F = 1$ and $C_A = 0$ in the QCD splitting functions [46, 47] in accordance with the gauge group U(1).

The *first order terms* are

$$\begin{aligned} P_{11}^{(1)}(x) &= P_{11\rm NS}^{(1)}(x) + P_{11\rm PS}^{(1)}(x) \\ &= \frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\delta(1-x)\,, \end{aligned} \tag{19}$$

$$P_{11\rm PS}^{(1)}(x) = 0\,, \tag{20}$$

$$P_{12}^{(1)}(x) = 2[x^2 + (1-x)^2] = 4x^2 - 4x + 2\,, \tag{21}$$

$$P_{21}^{(1)}(x) = \frac{1+(1-x)^2}{x} = x - 2 + \frac{2}{x}\,, \tag{22}$$

$$P_{22}^{(2)}(x) = -\frac{2}{3}\delta(1-x)\,. \tag{23}$$

The $\delta(1-x)$ distribution in (23) emerges due to momentum conservation for the photon momentum, $\int_0^1 \mathrm{d}x\, x[P_{12}(x) + P_{22}(x)] = 1$.

The *second order terms* are

$$\begin{aligned} P_{11}^{(2)}(x) &= P_{11\rm NS}^{(2)}(x) + P_{11\rm PS}^{(2)}(x) \\ &= 8\left(\frac{\ln(1-x)}{1-x}\right)_+ - 4(1+x)\ln(1-x) \\ &\quad + \ln(x)\left[7 + 7x - \frac{4}{1-x}\right] + \frac{6}{(1-x)_+} - 3 - 3x \\ &\quad - \frac{8}{3}x^2 + \frac{8}{3x} + \left[\frac{9}{4} - 4\zeta(2)\right]\delta(1-x)\,, \end{aligned} \tag{24}$$

$$\begin{aligned} P_{12}^{(2)}(x) &= 4(1-2x+2x^2)\ln(1-x) - 2(1-2x+4x^2)\ln(x) \\ &\quad - \frac{7}{3} + \frac{20}{3}x - \frac{8}{3}x^2, \end{aligned} \tag{25}$$

$$\begin{aligned} P_{21}^{(2)}(x) &= 2\ln(1-x)\left[x - 2 + \frac{2}{x}\right] + (2-x)\ln(x) \\ &\quad + \frac{10}{3} - \frac{7}{6}x - \frac{4}{3x}\,, \end{aligned} \tag{26}$$

$$P_{22}^{(2)}(x) = 4(1+x)\ln(x) + 2 - 2x - \frac{8}{3}x^2 + \frac{8}{3x} + \frac{4}{9}\delta(1-x)\,. \tag{27}$$

They were derived in [18] and various other places before.

The *third order terms* are

$$\begin{aligned} P_{11}^{(3)}(x) &= P_{11\rm NS}^{(3)}(x) + P_{11\rm PS}^{(3)}(x) \\ &= 24\left(\frac{\ln^2(1-x)}{1-x}\right)_+ - 12(1+x)\ln^2(1-x) \\ &\quad + 36\left(\frac{\ln(1-x)}{1-x}\right)_+ \\ &\quad - \ln(1-x)\left[22 + 14x + \frac{32}{3}x^2 - \frac{32}{3x}\right] \\ &\quad - 24\ln(x)\frac{\ln(1-x)}{1-x} + 34(1+x)\ln(x)\ln(1-x) \\ &\quad + \ln^2(x)\left[\frac{4}{1-x} - \frac{15}{2} - \frac{15}{2}x\right] \\ &\quad + \ln(x)\left[\frac{83}{6} + \frac{59}{6}x + \frac{32}{3}x^2 - \frac{18}{1-x}\right] \\ &\quad + 22(1+x)\mathrm{Li}_2(1-x) + \left[\frac{27}{2} - 24\zeta(2)\right]\frac{1}{(1-x)_+} \\ &\quad - \frac{317}{12} + \frac{155}{12}x + \frac{16}{9}x^2 - \frac{16}{9x} + 12(1+x)\zeta(2) \\ &\quad + \left[\frac{27}{8} - 18\zeta(2) + 16\zeta(3)\right]\delta(1-x)\,, \end{aligned} \tag{28}$$

$$\begin{aligned} P_{12}^{(3)}(x) &= 8(1-2x+2x^2)\ln^2(1-x) \\ &\quad - \frac{4}{3}(5 - 16x + 4x^2)\ln(1-x) \\ &\quad - 8(1-2x+4x^2)\ln(x)\ln(1-x) \\ &\quad - (3 - 6x - 8x^2)\ln^2(x) \\ &\quad - \ln(x)\left[\frac{32}{3} + \frac{26}{3}x - 16x^2\right] - 16x^2\mathrm{Li}_2(1-x) \\ &\quad - \frac{521}{18} + \frac{491}{9}x - \frac{232}{9}x^2 + \frac{32}{9x} \\ &\quad - 8(1-2x+2x^2)\zeta(2)\,, \end{aligned} \tag{29}$$

$$\begin{aligned} P_{21}^{(3)}(x) &= 4\left[x - 2 + \frac{2}{x}\right]\ln^2(1-x) \\ &\quad - \frac{2}{3}\left[5x - 16 + \frac{4}{x}\right]\ln(1-x) \\ &\quad + 4(2-x)\ln(x)\ln(1-x) \\ &\quad + \frac{3}{2}(2-x)\ln^2(x) + \frac{1}{3}\left(26x - 19 - \frac{16}{x}\right)\ln(x) \\ &\quad + \frac{8}{x}\mathrm{Li}_2(1-x) - 4\left[x - 2 + \frac{2}{x}\right]\zeta(2) \\ &\quad + \frac{491}{18} - \frac{521}{36}x + \frac{16}{9}x^2 - \frac{116}{9x}\,, \end{aligned} \tag{30}$$

$$\begin{aligned} P_{22}^{(3)}(x) &= 4\ln(1-x)\left[1 - x - \frac{4}{3}x^2 + \frac{4}{3}\frac{1}{x}\right] \\ &\quad + 8(1+x)\ln(x)\ln(1-x) - 2(1+x)\ln^2(x) \\ &\quad - \frac{4}{3}(4 + x - 4x^2)\ln(x) + 8(1+x)\mathrm{Li}_2(1-x) \\ &\quad - \frac{31}{3}(1-x) + \frac{32}{9}\left[x^2 - \frac{1}{x}\right] - \frac{8}{27}\delta(1-x)\,. \end{aligned} \tag{31}$$

The *fourth order terms* are as given below (here and for the fifth order terms we refer to the expressions $P_{\rm NS}^{(k)}(x)$ given in [26] for brevity):

$$
\begin{aligned}
P_{11}^{(4)}(x) = {} & P_{\rm NS}^{(4)}(x) + 48(1+x)\ln(x)\ln^2(1-x) \\
& - 24(1+x)\ln^2(x)\ln(1-x) - \frac{2}{3}(1+x)\ln^3(x) \\
& + 8\left[3 - 3x - 4x^2 + \frac{4}{x}\right]\ln^2(1-x) \\
& - 8\ln(x)\ln(1-x)\left[\frac{4}{3} - \frac{14}{3}x - 8x^2\right] \\
& + \ln^2(x)\left[-\frac{25}{3} + \frac{5}{3}x - 16x^2\right] \\
& - 4\left[\frac{73}{3} - \frac{73}{3}x - \frac{16}{9}x^2 + \frac{16}{9x}\right]\ln(1-x) \\
& - \left[\frac{569}{9} + \frac{1445}{9}x + \frac{128}{9}x^2 + \frac{64}{9x}\right. \\
& \quad \left. + 48\zeta(2) + 48\zeta(2)x\right]\ln(x) \\
& + 96(1+x)\ln(1-x)\mathrm{Li}_2(1-x) \\
& + 8\left[\frac{5}{3} + \frac{5}{3}x + 4x^2 + \frac{4}{x}\right]\mathrm{Li}_2(1-x) \\
& - 96(1+x)\mathrm{Li}_3(1-x) + 48(1+x)\mathrm{S}_{1,2}(1-x) \\
& - \frac{745}{18} + \frac{745}{18}x + \frac{224}{9}x^2 - \frac{224}{9x} \\
& - 8\left[3 - 3x - 4x^2 + \frac{4}{x}\right]\zeta(2)\,, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
P_{12}^{(4)} = {} & 16(1 - 2x + 2x^2)\ln^3(1-x) \\
& - 24(1 - 2x + 4x^2)\ln(x)\ln^2(1-x) \\
& - 2\ln^2(x)\ln(1-x)(5 - 10x - 24x^2) \\
& + \ln^3(x)\left[\frac{7}{3} - \frac{14}{3}x - \frac{16}{3}x^2\right] \\
& - 4\ln^2(1-x)\left[\frac{13}{3} - \frac{44}{3}x + \frac{8}{3}x^2\right] \\
& - 4\ln(x)\ln(1-x)\left[\frac{32}{3} + \frac{35}{3}x - 16x^2\right] \\
& + \ln^2(x)\left[\frac{79}{6} + \frac{14}{3}x - \frac{80}{3}x^2\right] \\
& - \ln(1-x)\left[\frac{959}{9} - \frac{1978}{9}x + \frac{320}{3}x^2 - \frac{128}{9x}\right. \\
& \qquad \left. + 48\zeta(2) - 96\zeta(2)x + 96\zeta(2)x^2\right] \\
& + \ln(x)\left[\frac{1505}{18} - \frac{614}{9}x + \frac{832}{9}x^2\right. \\
& \qquad \left. + 24\zeta(2) - 48\zeta(2)x + 96\zeta(2)x^2\right] \\
& - 96x^2\ln(1-x)\mathrm{Li}_2(1-x) \\
& - 44(1-2x)\ln(x)\mathrm{Li}_2(1-x) \\
& - 4\mathrm{Li}_2(1-x)\left[15 - 3x - \frac{40}{3}x^2\right] + 96x^2\mathrm{Li}_3(1-x) \\
& - 4(13 - 26x + 16x^2)\mathrm{S}_{1,2}(1-x) \\
& + \frac{18065}{108} - \frac{5947}{27}x + \frac{560}{9}x^2 - \frac{128}{27x} \\
& + 4\left[\frac{13}{3} - \frac{44}{3} + \frac{8}{3}x^2\right]\zeta(2) + 32(1 - 2x + 2x^2)\zeta(3)\,, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
P_{21}^{(4)} = {} & 8\left[x - 2 + \frac{2}{x}\right]\ln^3(1-x) \\
& + 12(2-x)\ln(x)\ln^2(1-x) \\
& + 5(2-x)\ln^2(x)\ln(1-x) - \frac{7}{6}(2-x)\ln^3(x) \\
& + \frac{2}{3}\ln^2(1-x)\left[44 - 13x - \frac{8}{x}\right] \\
& - 2\ln(x)\ln(1-x)\left[\frac{53}{3} - \frac{58}{3}x + \frac{32}{3}\frac{1}{x}\right] \\
& + \ln^2(x)\left[\frac{16}{3} - \frac{101}{12}x\right] \\
& + \ln(1-x)\left[\frac{989}{9} - \frac{959}{18}x + \frac{64}{9}x^2 - \frac{160}{3x}\right. \\
& \qquad \left. + 48\zeta(2) - 24\zeta(2)x - \frac{48}{x}\zeta(2)\right] \\
& + \ln(x)\left[-\frac{682}{9} + \frac{413}{36}x - \frac{64}{9}x^2 + \frac{64}{9x}\right. \\
& \qquad \left. - 24\zeta(2) + 12\zeta(2)x\right] \\
& + \frac{48}{x}\ln(1-x)\mathrm{Li}_2(1-x) \\
& + 22(2-x)\ln(x)\mathrm{Li}_2(1-x) \\
& - 2\mathrm{Li}_2(1-x)\left[3 - 15x + \frac{40}{3x}\right] - \frac{48}{x}\mathrm{Li}_3(1-x) \\
& + 2\left[26 - 13x + \frac{8}{x}\right]\mathrm{S}_{1,2}(1-x) \\
& - \frac{5947}{54} + \frac{18065}{216}x - \frac{64}{27}x^2 + \frac{280}{9x} \\
& - \frac{2}{3}\left[44 - 13x - \frac{8}{x}\right]\zeta(2) - 16\left[2 - x - \frac{2}{x}\right]\zeta(3)\,, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
P_{22}^{(4)}(x) = {} & 16(1+x)\ln(x)\ln^2(1-x) \\
& - 8(1+x)\ln^2(x)\ln(1-x) - 2(1+x)\ln^3(x) \\
& + 8\left[1 - x - \frac{4}{3}x^2 + \frac{4}{3x}\right]\ln^2(1-x) \\
& - \frac{16}{3}(2 - x - 4x^2)\ln(x)\ln(1-x) \\
& - \frac{1}{3}(19 - 23x + 16x^2)\ln^2(x) \\
& - 4\ln(1-x)\left[9 - 9x - \frac{16}{9}x^2 + \frac{16}{9x}\right] \\
& - \ln(x)\left[\frac{217}{3} + \frac{325}{3}x + \frac{128}{9}x^2 + \frac{64}{9x}\right. \\
& \qquad \left. + 16\zeta(2) + 16\zeta(2)x\right] \\
& + 32(1+x)\ln(1-x)\mathrm{Li}_2(1-x) \\
& - \frac{8}{3}\mathrm{Li}_2(1-x)\left[1 + x - 4x^2 - \frac{4}{x}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- 32(1+x)\mathrm{Li}_3(1-x) + 16(1+x)\mathrm{S}_{1,2}(1-x) \\
&- \frac{1331}{18} + \frac{1331}{18}x + \frac{608}{27}x^2 - \frac{608}{27x} \\
&- 8\left[1 - x - \frac{4}{3}x^2 + \frac{4}{3x}\right]\zeta(2) + \frac{16}{81}\delta(1-x)\,. \qquad (35)
\end{aligned}
$$

The *fifth order terms* are

$$
\begin{aligned}
P_{11}^{(5)}(x) = {}& P_{\mathrm{NS}}^{(5)}(x) + 128(1+x)\ln(x)\ln^3(1-x) \\
&- 96(1+x)\ln^2(x)\ln^2(1-x) + \frac{4}{3}(1+x)\ln^4(x) \\
&+ 64\left[1 - x - \frac{4}{3}x^2 + \frac{4}{3x}\right]\ln^3(1-x) \\
&- 32(1-5x-8x^2)\ln(x)\ln^2(1-x) \\
&- 8(7+x+16x^2)\ln^2(x)\ln(1-x) \\
&+ \frac{8}{9}\left[10 - 5x + 16x^2\right]\ln^3(x) \\
&- 64\left[6 - 6x - \frac{x^2}{3} + \frac{1}{3x}\right]\ln^2(1-x) \\
&- 32\ln(x)\ln(1-x)\left[\frac{97}{9} + \frac{313}{9}x + \frac{8}{3}x^2 + \frac{4}{3x} \right. \\
&\qquad\qquad \left. + 12\zeta(2) + 12\zeta(2)x\right] \\
&+ 16\ln^2(x)\left[\frac{83}{9} + \frac{122}{9}x + 2x^2 + 6\zeta(2) + 6\zeta(2)x\right] \\
&- 4\ln(1-x)\left[\frac{331}{9} - \frac{331}{9}x - \frac{1024}{27}x^2 + \frac{1024}{27x} \right. \\
&\qquad\qquad \left. + 48\zeta(2) - 48\zeta(2)x - 64\zeta(2)x^2 + \frac{64}{x}\zeta(2)\right] \\
&+ 4\ln(x)\left[\frac{1990}{27} + \frac{997}{27}x - \frac{320}{9}x^2 + \frac{64}{27x} + 8\zeta(2) \right. \\
&\qquad\qquad \left. - 40\zeta(2)x - 64\zeta(2)x^2 + 64\zeta(3) + 64\zeta(3)x\right] \\
&+ 384(1+x)\ln^2(1-x)\mathrm{Li}_2(1-x) \\
&- 96(1+x)\ln^2(x)\mathrm{Li}_2(1-x) \\
&+ 128\left[1 + x + 2x^2 + \frac{2}{x}\right]\ln(1-x)\mathrm{Li}_2(1-x) \\
&- 144(1-x)\ln(x)\mathrm{Li}_2(1-x) \\
&- 768(1+x)\ln(1-x)\mathrm{Li}_3(1-x) \\
&+ 384(1+x)\ln(1-x)\mathrm{S}_{1,2}(1-x) \\
&- 256(1+x)\ln(x)\mathrm{S}_{1,2}(1-x) \\
&- 32\left[\frac{205}{9} + \frac{205}{9}x + 2x^2 + \frac{2}{x} \right. \\
&\qquad \left. + 12\zeta(2) + 12\zeta(2)x\right]\mathrm{Li}_2(1-x) \\
&- 128\left[1 + x + 2x^2 + \frac{2}{x}\right]\mathrm{Li}_3(1-x) \\
&- 16\left[7 - 15x - \frac{32}{3}x^2 - \frac{16}{3x}\right]\mathrm{S}_{1,2}(1-x) \\
&+ 768(1+x)\mathrm{Li}_4(1-x) - 192(1+x)\mathrm{S}_{1,3}(1-x) \\
&- 384(1+x)\mathrm{S}_{2,2}(1-x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{27}\left[10127 - 10127x - \frac{2456}{3}x^2 + \frac{2456}{3x}\right] \\
&+ 64\left[6 - 6x - \frac{1}{3}x^2 + \frac{1}{3x}\right]\zeta(2) \\
&+ 128\left[1 - x - \frac{4}{3}x^2 + \frac{4}{3x}\right]\zeta(3)\,, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
P_{12}^{(5)}(x) = {}& 32(1-2x+2x^2)\ln^4(1-x) \\
&- 64(1-2x+4x^2)\ln(x)\ln^3(1-x) \\
&- 24(1-2x-8x^2)\ln^2(x)\ln^2(1-x) \\
&+ \frac{8}{3}(5-10x-16x^2)\ln^3(x)\ln(1-x) \\
&+ \ln^4(x)\left[\frac{5}{12} - \frac{5}{6}x + \frac{8}{3}x^2\right] \\
&- \frac{64}{3}(2-7x+x^2)\ln^3(1-x) \\
&- 16(8+11x-12x^2)\ln(x)\ln^2(1-x) \\
&+ \frac{8}{3}(26+23x-60x^2)\ln^2(x)\ln(1-x) \\
&+ \ln^3(x)\left[5 - \frac{x}{3} + \frac{224}{9}x^2\right] \\
&- 4\ln^2(1-x)\left[\frac{649}{9} - \frac{1478}{9}x + \frac{728}{9}x^2 - \frac{32}{3x} \right. \\
&\qquad\qquad \left. + 48\zeta(2) - 96\zeta(2)x + 96\zeta(2)x^2\right] \\
&+ 4\ln(x)\ln(1-x)\left[\frac{1027}{9} - \frac{866}{9}x + \frac{1328}{9}x^2 \right. \\
&\qquad\qquad \left. + 48\zeta(2) - 96\zeta(2)x + 192\zeta(2)x^2\right] \\
&+ \ln^2(x)\left[\frac{451}{9} - \frac{1664}{9}x - \frac{1072}{9}x^2 + 24\zeta(2) \right. \\
&\qquad\qquad \left. - 48\zeta(2)x - 192\zeta(2)x^2\right] \\
&+ 4\ln(1-x)\left[\frac{6064}{27} - \frac{7721}{27}x + \frac{2056}{27}x^2 - \frac{128}{27x} \right. \\
&\qquad\qquad + 32\zeta(2) - 112\zeta(2)x + 16\zeta(2)x^2 \\
&\qquad\qquad \left. + 64\zeta(3) - 128\zeta(3)x + 128\zeta(3)x^2\right] \\
&+ \ln(x)\left[\frac{8861}{54} + \frac{24511}{27}x - 416x^2 - \frac{256}{27x} \right. \\
&\qquad\qquad + 128\zeta(2) + 176\zeta(2)x - 192\zeta(2)x^2 \\
&\qquad\qquad \left. - 128\zeta(3) + 256\zeta(3)x - 512\zeta(3)x^2\right] \\
&- 384x^2\ln^2(1-x)\mathrm{Li}_2(1-x) \\
&+ 288(2x-1)\ln(x)\ln(1-x)\mathrm{Li}_2(1-x) \\
&+ 16(1-2x+4x^2)\ln^2(x)\mathrm{Li}_2(1-x) \\
&- 32(12-3x-10x^2)\ln(1-x)\mathrm{Li}_2(1-x) \\
&+ \frac{32}{3}(1-5x-12x^2)\ln(x)\mathrm{Li}_2(1-x) \\
&+ 768x^2\ln(1-x)\mathrm{Li}_3(1-x) \\
&- 288(2x-1)\ln(x)\mathrm{Li}_3(1-x) \\
&- 32(11-22x+16x^2)\ln(1-x)\mathrm{S}_{1,2}(1-x) \\
&- 16(5-10x-16x^2)\ln(x)\mathrm{S}_{1,2}(1-x)
\end{aligned}
$$

$$
\begin{aligned}
&+8\left[21+34x+\frac{100}{3}x^2+\frac{16}{3x}+48\zeta(2)x^2\right]\\
&\times \mathrm{Li}_2(1-x)\\
&+32(12-3x-10x^2)\mathrm{Li}_3(1-x)\\
&-16\left[10+5x-\frac{8}{3}x^2\right]\mathrm{S}_{1,2}(1-x)\\
&-768x^2\mathrm{Li}_4(1-x)+32(1-2x+2x^2)\mathrm{Li}_2^2(1-x)\\
&-32(7-14x-6x^2)\mathrm{S}_{1,3}(1-x)\\
&+32(7-14x+8x^2)\mathrm{S}_{2,2}(1-x)\\
&+\frac{322519}{648}-\frac{240997}{324}x+\frac{23776}{81}x^2-\frac{1088}{27x}\\
&+\frac{4}{3}\left[\frac{649}{3}-\frac{1478}{3}x+\frac{728}{3}x^2-\frac{32}{x}\right]\zeta(2)\\
&-\frac{128}{3}(2-7x+x^2)\zeta(3)+48(1-2x+2x^2)\zeta(4)\,,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
P_{21}^{(5)}(x) = &-16\left[2-x-\frac{2}{x}\right]\ln^4(1-x)\\
&+32(2-x)\ln(x)\ln^3(1-x)\\
&+12(2-x)\ln^2(x)\ln^2(1-x)\\
&-\frac{20}{3}(2-x)\ln^3(x)\ln(1-x)-\frac{5}{24}(2-x)\ln^4(x)\\
&+\frac{32}{3}\left[7-2x-\frac{1}{x}\right]\ln^3(1-x)\\
&-8\left[17-16x+\frac{8}{x}\right]\ln(x)\ln^2(1-x)\\
&+\frac{4}{3}(41-46x)\ln^2(x)\ln(1-x)+\frac{1}{6}(x-47)\ln^3(x)\\
&+\ln^2(1-x)\left[\frac{2956}{9}-\frac{1298}{9}x+\frac{64}{3}x^2-\frac{1456}{9x}\right.\\
&\qquad\left.+192\zeta(2)-96\zeta(2)x-\frac{192}{x}\zeta(2)\right]\\
&-\ln(x)\ln(1-x)\left[\frac{4180}{9}-\frac{542}{9}x+\frac{128}{3}x^2-\frac{256}{9x}\right.\\
&\qquad\left.+192\zeta(2)-96\zeta(2)x\right]\\
&-\ln^2(x)\left[\frac{220}{9}-\frac{1207}{18}x-\frac{32}{3}x^2-\frac{64}{9x}\right.\\
&\qquad\left.+24\zeta(2)-12\zeta(2)x\right]\\
&-\ln(1-x)\left[\frac{15442}{27}-\frac{12128}{27}x+\frac{256}{27}x^2-\frac{4112}{27x}\right.\\
&\qquad+224\zeta(2)-64\zeta(2)x-\frac{32}{x}\zeta(2)\\
&\qquad\left.+256\zeta(3)-128\zeta(3)x-\frac{256}{x}\zeta(3)\right]\\
&+\ln(x)\left[\frac{6373}{54}-\frac{57373}{108}x+\frac{128}{9}x^2+\frac{1504}{27x}\right.\\
&\qquad+136\zeta(2)-128\zeta(2)x+\frac{64}{x}\zeta(2)\\
&\qquad\left.+128\zeta(3)-64\zeta(3)x\right]\\
&+\frac{192}{x}\ln^2(1-x)\mathrm{Li}_2(1-x)\\
&+144(2-x)\ln(x)\ln(1-x)\mathrm{Li}_2(1-x)\\
&-8(2-x)\ln^2(x)\mathrm{Li}_2(1-x)\\
&-16\left[3-12x+\frac{10}{x}\right]\ln(1-x)\mathrm{Li}_2(1-x)\\
&-\frac{16}{3}\left[5-x+\frac{12}{x}\right]\ln(x)\mathrm{Li}_2(1-x)\\
&-\frac{384}{x}\ln(1-x)\mathrm{Li}_3(1-x)\\
&-144(2-x)\ln(x)\mathrm{Li}_3(1-x)\\
&+16\left[22-11x+\frac{8}{x}\right]\ln(1-x)\mathrm{S}_{1,2}(1-x)\\
&+40(2-x)\ln(x)\mathrm{S}_{1,2}(1-x)\\
&-4\left[34+21x+\frac{16}{3}x^2+\frac{100}{3x}+\frac{48}{x}\zeta(2)\right]\mathrm{Li}_2(1-x)\\
&+16\left[3-12x+\frac{10}{x}\right]\mathrm{Li}_3(1-x)\\
&-8\left[11-14x+\frac{52}{3x}\right]\mathrm{S}_{1,2}(1-x)\\
&+\frac{384}{x}\mathrm{Li}_4(1-x)-16\left[2-x-\frac{2}{x}\right]\mathrm{Li}_2^2(1-x)\\
&+16\left[14-7x+\frac{2}{x}\right]\mathrm{S}_{1,3}(1-x)\\
&-16\left[14-7x+\frac{16}{x}\right]\mathrm{S}_{2,2}(1-x)\\
&-\frac{240997}{648}+\frac{322519}{1296}x-\frac{544}{27}x^2+\frac{11888}{81x}\\
&-\left[\frac{2956}{9}-\frac{1298}{9}x+\frac{64}{3}x^2-\frac{1456}{9x}\right]\zeta(2)\\
&+\frac{64}{3}\left[7-2x-\frac{1}{x}\right]\zeta(3)-24\left[2-x-\frac{2}{x}\right]\zeta(4)\,,
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
P_{22}^{(5)}(x) = &\,32(1+x)\ln(x)\ln^3(1-x)\\
&-24(1+x)\ln^2(x)\ln^2(1-x)\\
&-\frac{20}{3}(1+x)\ln^3(x)\ln(1-x)+\frac{7}{6}(1+x)\ln^4(x)\\
&+16\left[1-x-\frac{4}{3}x^2+\frac{4}{3x}\right]\ln^3(1-x)\\
&-\frac{16}{3}(4-5x+12x^2)\ln(x)\ln^2(1-x)\\
&-\ln^2(x)\ln(1-x)\left[\frac{82}{3}-\frac{74}{3}x+32x^2\right]\\
&+\frac{2}{9}\ln^3(x)\left[35-10x+16x^2\right]\\
&-\frac{4}{3}\left[77-77x-\frac{32}{3}x^2+\frac{32}{3x}\right]\ln^2(1-x)\\
&-\ln(x)\ln(1-x)\left[\frac{862}{3}+\frac{1478}{3}x+\frac{512}{9}x^2+\frac{256}{9x}\right.\\
&\qquad\left.+96\zeta(2)+96\zeta(2)x\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\ln^2(x)\left[\frac{319}{3}+\frac{266}{3}x+\frac{64}{3}x^2\right.\\
&\qquad\left.+24\zeta(2)+24\zeta(2)x\right]\\
&-\ln(1-x)\left[\frac{2401}{9}-\frac{2401}{9}x-\frac{2624}{27}x^2+\frac{2624}{27x}\right.\\
&\qquad\left.+48\zeta(2)-48\zeta(2)x-64\zeta(2)x^2+\frac{64}{x}\zeta(2)\right]\\
&+\ln(x)\left[\frac{8680}{27}+\frac{1477}{27}x-\frac{2240}{27}x^2+\frac{128}{9x}\right.\\
&\qquad+\frac{64}{3}\zeta(2)-\frac{80}{3}\zeta(2)x-64\zeta(2)x^2\\
&\qquad\left.+64\zeta(3)+64\zeta(3)x\right]\\
&+96(1+x)\ln^2(1-x)\mathrm{Li}_2(1-x)\\
&-44(1+x)\ln^2(x)\mathrm{Li}_2(1-x)\\
&+\frac{16}{3}\left[1+x+12x^2+\frac{12}{x}\right]\ln(1-x)\mathrm{Li}_2(1-x)\\
&-76(1-x)\ln(x)\mathrm{Li}_2(1-x)\\
&-192(1+x)\ln(1-x)\mathrm{Li}_3(1-x)\\
&+96(1+x)\ln(1-x)\mathrm{S}_{1,2}(1-x)\\
&-104(1+x)\ln(x)\mathrm{S}_{1,2}(1-x)\\
&-\mathrm{Li}_2(1-x)\left[390+390x+\frac{128}{3}x^2+\frac{128}{3x}\right.\\
&\qquad\left.+96\zeta(2)+96\zeta(2)x\right]\\
&-\frac{16}{3}\left[1+x+12x^2+\frac{12}{x}\right]\mathrm{Li}_3(1-x)\\
&-\frac{4}{3}\left[61-65x-32x^2-\frac{16}{x}\right]\mathrm{S}_{1,2}(1-x)\\
&+192(1+x)\mathrm{Li}_4(1-x)-88(1+x)\mathrm{S}_{1,3}(1-x)\\
&-96(1+x)\mathrm{S}_{2,2}(1-x)\\
&+\frac{24043}{36}-\frac{24043}{36}x-\frac{5408}{81}x^2+\frac{5408}{81x}\\
&+\frac{4}{3}\left[77-77x-\frac{32}{3}x^2+\frac{32}{3x}\right]\zeta(2)\\
&+32\left[1-x-\frac{4}{3}x^2+\frac{4}{3x}\right]\zeta(3)-\frac{32}{243}\delta(1-x)\,.
\end{aligned}
\tag{39}
$$

The complexity of the above expressions reaches weight $w=n+p=4$ Nielsen integrals

$$
S_{n,p}(x)=\frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1\frac{\mathrm{d}z}{z}\ln^{n-1}(z)\ln^p(1-zx)\,. \tag{40}
$$

The radiators can be expressed through these functions and polynomials thereof as well as rational functions in x. Nielsen integrals form a sub-class of harmonic polylogarithms [48]. However, in the present case, this generalization is not needed since *all* convolutions required remain in this sub-class. This is due to the fact that the respective concatenation products of the index set [49, 50] contain only the 'letters' x_0 and x_1, which stand for the Poincaré–Chen integral iteration of $1/x$ and $1/(1-x)$, respectively. The letter x_{-1} corresponding to the integral iteration of $1/(1+x)$ does not occur. It is only the latter one that extends the class of Nielsen integrals to those dealt with in [48]. Precise numerical representations of polylogarithms and Nielsen integrals are given in [51].

The universal radiator functions (14) can now be attached to the respective initial- or final-state radiating light charged fermion or photon lines of any differential scattering cross section to account for the respective leading order QED corrections. These radiators generalize the radiators due to soft-photon exponentiation, valid for $D_{\mathrm{NS}}(a(Q^2),x)$ [53, 54] in the region $x\to 1$, to general values of x and all collinear transitions possible. The numerical effect of the respective radiator depends on the change of the subsystem kinematics of the differential scattering cross section, which usually differs for initial- and final-state radiation, and is due to the type of leg encountered. This kinematics has to be worked out for the respective process accordingly. Moreover, the radiative corrections may strongly depend on the way the kinematic variables of the process are measured. In case of deeply inelastic scattering, investigations of these aspects were performed in [12, 19, 20, 52]. For similar considerations for e^+e^- annihilation see e.g. [11].

The radiative correction due to the radiator $D_{a_1a_2}(a(Q^2),x)$ for a differential cross section reads

$$
\begin{aligned}
\frac{\mathrm{d}^l\sigma_{a_1}}{\mathrm{d}b_1\dots\mathrm{d}b_l}&=\int_0^1\mathrm{d}zD_{a_1a_2}(a(Q^2),z)\theta\left(z-z_0^{a_1}\right)\\
&\quad\times J^{a_1}\left(b_r|_{r=1}^l,z\right)\frac{\mathrm{d}^l\sigma_{a_2}}{\mathrm{d}b_1\dots\mathrm{d}b_l}\bigg|_{b_1=\hat b_1\dots b_l=\hat b_l}\,.
\end{aligned}
\tag{41}
$$

Here the l kinematic variables that determine the differential cross section are $b_r|_{r=1}^l$. Their rescaled values under changing the momentum $p_{a_1}\to z\cdot p_{a_1}$ respectively $p_{a_1}\to p_{a_1}/z$ for initial- or final-state radiation, $\hat b_r|_{r=1}^l$, are bounded by $z_0^{a_1}$ for hard radiation. $J^{a_1}(b_r|_{r=1}^l,z)$ denotes the corresponding Jacobian

$$
J^{a_1}(b_r|_{r=1}^l,z)=\begin{vmatrix}\partial\hat b_l/\partial b_1 & \cdots & \partial\hat b_l/\partial b_1\\ \vdots & & \vdots\\ \partial\hat b_l/\partial b_1 & \cdots & \partial\hat b_l/\partial b_l\end{vmatrix} \tag{42}
$$

and $\mathrm{d}^l\sigma_{a_2}/\mathrm{d}b_1\dots\mathrm{d}b_l$ is the subsystem differential cross section for which the line of type a_1 is being replaced by a line of type a_2. Equation (41) may be generalized to the case of more universal radiators correspondingly, requiring additional rescaling of variables.

In the above, we assumed that the radiator functions describe collinear radiation along outer fermions or photons. However, in various applications also *internal*, nearly collinear situations may occur. One example is the so-called third [55] or Compton peak [11, 12, 56] in deeply inelastic scattering. Here a photon being originally virtual contributes close to its mass shell in the radiative correction, which gives rise to a factorizing collinear process. The universal contributions to these processes can be obtained from radiators as well.

Finally we would like to add a remark on small-x resummations. In QCD the leading order corrections stem from

$$P_{gg}^{x\to 0}(x) = \mathbf{M}^{-1}[\gamma_L(N, a_{\rm s})](x)\,, \tag{43}$$

$$P_{gq}^{x\to 0}(x) = \frac{C_F}{C_A} P_{gg}^{x\to 0}(x)\,, \tag{44}$$

and the function $\gamma_L(N, a_{\rm s})$ obeys

$$\gamma_L(N, a_{\rm s}) = \frac{C_A a_{\rm s}}{\pi(N-1)} \left\{ 1 + 2\sum_{l=1}^{\infty} \zeta_{2l+1} \gamma_L^{2l+1} \right\}, \tag{45}$$

with $a_{\rm s} = \alpha_{\rm s}/(4\pi)$ the strong coupling constant and ζ_k Riemann's ζ-function. The transition to QED, $C_A \to 0$, $C_F \to 1$, trivializes (43) and (44), except for the lowest order term in $a(Q^2)$ in $P_{\gamma f}^{x\to 0}(x)$, which is already known from the fixed order terms above. Yet all anomalous dimensions do receive $1/x$ terms in higher orders, which contribute, as well known [57–59], in the abelian limit starting from next-to-leading order:

$$P_{ff}^{(1)}(x) \propto \frac{1}{x}\frac{40}{9} N_f\,, \tag{46}$$

$$P_{\gamma f}^{(1)}(x) \propto \frac{1}{x}\frac{40}{9} N_f\,, \tag{47}$$

$$P_{f\gamma}^{(1)}(x) \propto \frac{1}{x}\frac{40}{9} N_f\,, \tag{48}$$

$$P_{\gamma\gamma}^{(1)}(x) \propto \frac{1}{x}\frac{4}{3} N_f\,. \tag{49}$$

However, these terms do not stem from the resummation [38] but are of different origin. Their pole strength is of $O(\alpha^2/(N-1))$, which is larger than for the poles resulting from $O(\alpha^2 \ln^2(x))$ terms. For unpolarized QED radiators systematic resummations of the leading small-x terms have not been carried out yet. At leading order in $a(Q^2)$ only $P_{\gamma f}(x) \propto 1/x$, while at next-to-leading order (46)–(49) all terms contain a singular contribution. It would be worthwhile to derive resummations of these terms in the future.

4 Numerical results

The singlet contributions to the universal radiator functions, summing the leading logarithmic corrections up to $O((\alpha L)^5)$, are shown in Fig. 1 as a function of the momentum fraction x for different values of $Q = \sqrt{Q^2}$ in the case of $m_f = m_e$. The corrections in case of other charged fermions have to be rescaled accordingly in $L = \ln(Q^2/m_f^2)$. In case of fractionally charged leptons, α_0 has to be replaced by $e_f^2\alpha_0$. Here $D_{11}^{\rm PS}$ denotes the pure singlet part of the fermion radiator to which the non-singlet contribution has to be added [26]. The diagonal radiator $D_{11}^{\rm PS}$ vanishes at leading order $O(\alpha L)$, while the radiator D_{22} contributes at $x = 1$ only due to momentum conservation. For $x < 1$ the (pure) singlet (PS)-diagonal radiators contribute with $O((\alpha L)^2)$ only. In this order the PS-diagonal

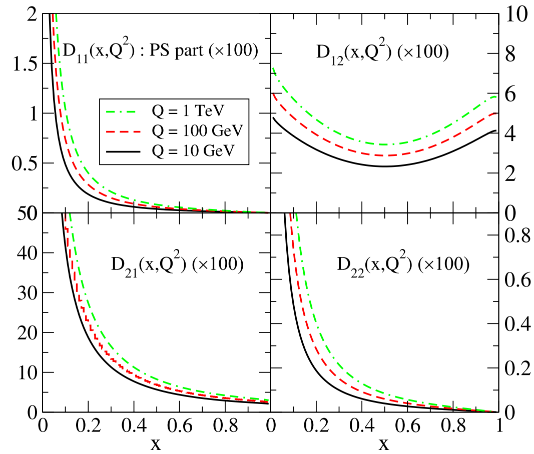

Fig. 1. The singlet radiators D_{ij} as a function of x and Q in %. $D_{11}^{\rm PS}$ denotes the pure singlet part of D_{11}

terms are identical in the region $x < 1$ and differ by the $\delta(1-x)$ distribution in D_{22}. At higher orders both diagonal elements receive different corrections. This explains the relative smallness of the radiators $D_{11}^{\rm PS}$ and D_{22} compared to D_{12} and D_{21}. Yet the diagonal radiators grow $\propto 1/x$ as $x \to 0$. This growth is even more pronounced in D_{21}, which contains a $1/x$ term already at $O(\alpha L)$, while D_{12} receives those terms at $O((\alpha L)^3)$ only and therefore shows a moderately varying profile in x. The QED scaling violations shown in Fig. 1 are of moderate size, comparing scales from $Q = 10$ GeV to $Q = 1$ TeV, which is due to the smallness of the fine structure constant and its weak running. At $x = 0.1$ the radiators $D_{11}^{\rm PS}$ and D_{22} reach about 1% and grow further towards smaller values of x. D_{12} takes values between 3 and 8%. D_{21} is largest and reaches 50% at $x = 0.1$. The radiators $D_{11}^{\rm PS}$, D_{21} and D_{22} vanish towards $x \to 1$, while D_{12} approaches finite values.

Figure 2 compares the size of the first order contribution to D_{11}, D_{12} and D_{21} with the respective contributions up to $O((\alpha L)^5)$, which can be regarded numerically as the total contribution for the values of Q chosen. For D_{11} the first order contribution is much smaller than the total contribution in the region of small values x due to the steep rise of the pure singlet component. The higher order contributions to D_{12} are small at medium values of x, and amount to a -5% correction in the small-x region and a $+10\%$ correction at large values of x for $Q = 10$ GeV. The higher order contribution to D_{21} range between 1% to 10%.

Figure 3 shows the impact of the fifth order term with respec to the first four orders. The effect of the fifth order term amounts to $O(10^{-5})$ for D_{11}, D_{12} and D_{21}, while the effect in the case of D_{22} is one order of magnitude larger at large value of x. In either case, we confirm that the singlet radiator to the fifth order has a very high accuracy, which is sufficient to represent the universal part of QED corrections, relevant in high precision measurements in both high-energy charged lepton–anti-lepton collisions, charged lepton–nucleon collisions and photon collisions at future electron–positron linear colliders with a possible Giga-Z option, electron–photon and photon–photon colliders and

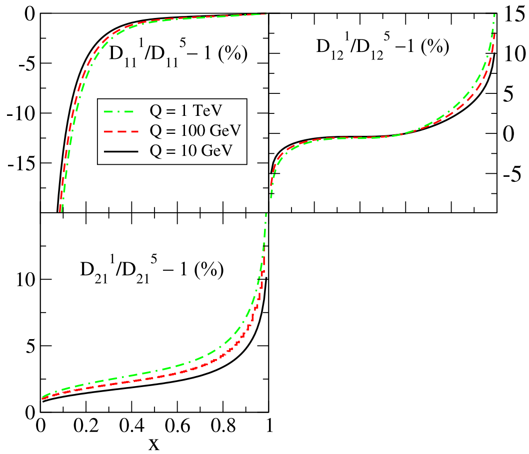

Fig. 2. Relative contribution of the first order singlet radiators D_{ij}^1 in all terms to $O((\alpha L)^5)$. Here D_{11} denotes the sum of the non-singlet contributions $D_{\rm NS}$, with soft exponentiation beyond $O((\alpha L)^5)$, and the pure singlet contribution $D_{11}^{\rm PS}$

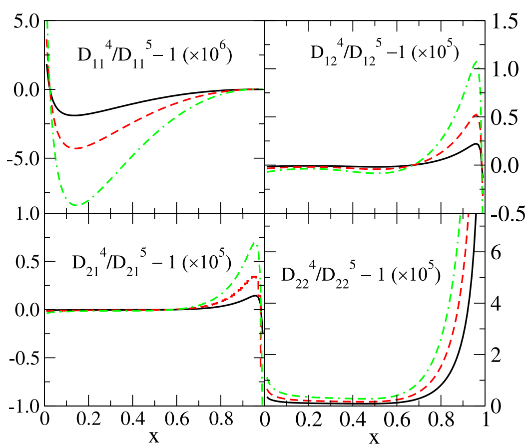

Fig. 3. Relative contribution of the singlet radiators D_{ij} up to the fourth order in αL if compared to all terms to the fifth order as a function of x and Q, with $D_{11} = D_{11}^{\rm NS+PS}$

future muon colliders. These are the reactions to which the corresponding radiators are contributing. The reactions do also contribute to the precision measurements of QCD scaling violations [12, 52, 60–62] in deeply inelastic scattering as universal QED corrections. Likewise they are important for rare initial states at high-energy hadron colliders such as LHC and would contribute in lepton and photon initiated processes there, such as single leptoquark production [63, 64].

The radiator functions calculated above are made available in the form of a FORTRAN program, which can be obtained from the authors upon request.

5 Conclusions

The collinear logarithms in QED can be resummed due to the renormalization group equations for mass factorization. Unlike the case in QCD, the collinear logarithms are finite due to the possibility to define the coupling constant asymptotically, i.e. in the limit of vanishing scales. The associated logarithms are well defined since photons and leptons are non-confined and may be regarded as stable or long-lived states. The leading order corrections $O((\alpha L)^k)$ are universal. The respective radiators resum the radiative corrections, which only depend on the type of particle transition $i \rightarrow j$. As shown, sufficient numerical stability of $O(10^{-4\ldots-5})$ is reached evaluating the radiators to $O((\alpha L)^5)$ for scales as large as $Q \lesssim 1$ TeV. The radiators are presented in x space and can be applied directly to the respective multiply differential scattering cross sections to describe the universal contribution due to initial- and final-state radiation off the different outer legs contributing to the respective scattering process involving charged fermions and photons. In the small-x region the leading order radiators receive contributions $\propto 1/x$, with an onset in different orders in αL, which leads to larger corrections in this kinematic region. A systematic resummation of the particular small-x contributions, unlike the case for the non-singlet and polarized singlet corrections, is not known yet. The radiators derived can easily be adopted for experimental analysis and simulation programs.

Appendix : Mellin convolutions

In this appendix we list the convolutions of functions up to weight 5 required in the present calculation in addition to those given in [26, 45]. Some of the integrals require one to use Mellin transforms and algebraic relations between the finite harmonic sums [50, 65]. They were calculated recursively in explicit form and may be of general interest for other higher order calculations in QED and QCD.

$$1 \otimes \frac{1}{x} = \frac{1}{x} - 1\,, \tag{A.1}$$

$$x \otimes \frac{1}{x} = \frac{1}{2}\left(\frac{1}{x} - x\right), \tag{A.2}$$

$$x^2 \otimes \frac{1}{x} = \frac{1}{3}\left(\frac{1}{x} - x^2\right), \tag{A.3}$$

$$\frac{1}{x} \otimes \frac{1}{x} = -\frac{1}{x}\ln(x)\,, \tag{A.4}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{1}{x} = \frac{1}{x}\ln(1-x)\,, \tag{A.5}$$

$$\left(\frac{\ln(1-x)}{1-x}\right)_+ \otimes \frac{1}{x} = \frac{1}{2x}\ln^2(1-x)\,, \tag{A.6}$$

$$\ln(1-x) \otimes \frac{1}{x} = \left(\frac{1}{x} - 1\right)[\ln(1-x) - 1]\,, \tag{A.7}$$

$$x\ln(1-x) \otimes \frac{1}{x} = \frac{1}{2}\left(\frac{1}{x} - x\right)\ln(1-x) + \frac{x}{4} + \frac{1}{2} - \frac{3}{4x}\,, \tag{A.8}$$

$$\ln(x) \otimes \frac{1}{x} = -\ln(x) + 1 - \frac{1}{x}\,, \tag{A.9}$$

$$x\ln(x) \otimes \frac{1}{x} = -\frac{x}{2}\ln(x) + \frac{1}{4}\left(x - \frac{1}{x}\right), \tag{A.10}$$

$$\frac{\ln(x)}{1-x}\otimes\frac{1}{x}=-\frac{1}{x}\mathrm{Li}_2(1-x)\,, \tag{A.11}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x^2\ln(1-x)$$
$$=x^2\left[\ln^2(1-x)-\ln(x)\ln(1-x)-\zeta_2\right]$$
$$+\frac{1}{2}(1+2x-3x^2)\ln(1-x)+\frac{5}{4}x^2-\frac{x}{2}-\frac{3}{4}\,, \tag{A.12}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x^2\ln(x)$$
$$=x^2\left[-\frac{1}{2}\ln^2(x)+\mathrm{Li}_2(1-x)+\ln(x)\ln(1-x)\right]$$
$$-\frac{3}{2}x^2\ln(x)-\frac{1}{4}-x+\frac{5}{4}x^2\,, \tag{A.13}$$

$$1\otimes x^2\ln(1-x)=\frac{1}{2}(1-x^2)\ln(1-x)+\frac{x^2}{4}+\frac{x}{2}-\frac{3}{4}\,, \tag{A.14}$$

$$1\otimes x^2\ln(x)=-\frac{x^2}{2}\ln(x)-\frac{1}{4}(1-x^2)\,, \tag{A.15}$$

$$x\otimes x^2\ln(1-x)=x(1-x)[\ln(1-x)-1]\,, \tag{A.16}$$

$$x\otimes x^2\ln(x)=-x[1-x+x\ln(x)]\,, \tag{A.17}$$

$$1\otimes x^2=\frac{1}{2}(1-x^2)\,, \tag{A.18}$$

$$x\otimes x^2=x(1-x)\,, \tag{A.19}$$

$$x^2\otimes x^2=-x^2\ln(x)\,, \tag{A.20}$$

$$x^2\otimes\ln(x)=\frac{1}{2}\ln(x)+\frac{1}{4}(1-x^2)\,, \tag{A.21}$$

$$x^2\otimes x\ln(x)=x[1-x+\ln(x)]\,, \tag{A.22}$$

$$\frac{1}{x}\ln(1-x)\otimes\frac{1}{x}=\frac{1}{x}[\mathrm{Li}_2(x)-\zeta_2]\,, \tag{A.23}$$

$$x^2\ln(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln(1-x)-\frac{11}{18x}+\frac{1}{3}+\frac{x}{6}+\frac{x^2}{9}\,, \tag{A.24}$$

$$x^2\ln(x)\otimes\frac{1}{x}=-\frac{x^2}{3}\ln(x)+\frac{1}{9}(x^2-\frac{1}{x})\,, \tag{A.25}$$

$$\left(\frac{\ln^2(1-x)}{1-x}\right)_+\otimes\frac{1}{x}=\frac{1}{3x}\ln^3(1-x)\,, \tag{A.26}$$

$$\ln^2(1-x)\otimes\frac{1}{x}=\left(\frac{1}{x}-1\right)[\ln^2(1-x)-2\ln(1-x)+2]\,, \tag{A.27}$$

$$x\ln^2(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{2}\left(\frac{1}{x}-x\right)\ln^2(1-x)+\left(\frac{x}{2}+1-\frac{3}{2x}\right)\ln(1-x)$$
$$+\frac{7}{4x}-\frac{3}{2}-\frac{x}{4}\,, \tag{A.28}$$

$$x^2\ln^2(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln^2(1-x)+\left(\frac{2}{9}x^2+\frac{x}{3}+\frac{2}{3}-\frac{11}{9x}\right)\ln(1-x)$$
$$+\frac{85}{54x}-\frac{11}{9}-\frac{5}{18}x-\frac{2}{27}x^2\,, \tag{A.29}$$

$$\frac{\ln(x)\ln(1-x)}{1-x}\otimes\frac{1}{x}=\frac{1}{x}[\mathrm{Li}_3(1-x)-\ln(1-x)\mathrm{Li}_2(1-x)]\,, \tag{A.30}$$

$$\ln(x)\ln(1-x)\otimes\frac{1}{x}$$
$$=-\frac{1}{x}\mathrm{Li}_2(1-x)-\ln(x)\ln(1-x)+\left(1-\frac{1}{x}\right)\ln(1-x)$$
$$+\ln(x)-2\left(1-\frac{1}{x}\right)\,, \tag{A.31}$$

$$x\ln(x)\ln(1-x)\otimes\frac{1}{x}$$
$$=-\frac{1}{2x}\mathrm{Li}_2(1-x)-\frac{x}{2}\ln(x)\ln(1-x)+\frac{1}{4}\left(x-\frac{1}{x}\right)\ln(1-x)$$
$$+\frac{1}{4}(2+x)\ln(x)+\frac{1}{x}-\frac{3}{4}-\frac{x}{4}\,, \tag{A.32}$$

$$x^2\ln(x)\ln(1-x)\otimes\frac{1}{x}$$
$$=-\frac{1}{3x}\mathrm{Li}_2(1-x)-\frac{x^2}{3}\ln(x)\ln(1-x)$$
$$+\frac{1}{9}\left(x^2-\frac{1}{x}\right)\ln(1-x)+\frac{1}{18}(6+3x+2x^2)\ln(x)$$
$$+\frac{71}{108x}-\frac{4}{9}-\frac{5}{36}x-\frac{2}{27}x^2\,, \tag{A.33}$$

$$\frac{\ln^2(x)}{1-x}\otimes\frac{1}{x}=\frac{2}{x}\mathrm{S}_{1,2}(1-x)\,, \tag{A.34}$$

$$\ln^2(x)\otimes\frac{1}{x}=\frac{2}{x}-2[1-\ln(x)]-\ln^2(x)\,, \tag{A.35}$$

$$x\ln^2(x)\otimes\frac{1}{x}=\frac{1}{4}\left[\frac{1}{x}-x\right]+\frac{x}{2}\left[\ln(x)-\ln^2(x)\right]\,, \tag{A.36}$$

$$x^2\ln^2(x)\otimes\frac{1}{x}=\frac{2}{27}\left[\frac{1}{x}-x^2\right]+x^2\left[\frac{2}{9}\ln(x)-\frac{1}{3}\ln^2(x)\right]\,, \tag{A.37}$$

$$\mathrm{Li}_2(1-x)\otimes\frac{1}{x}=\left(\frac{1}{x}-1\right)[\mathrm{Li}_2(1-x)-1]-\ln(x)\,, \tag{A.38}$$

$$x\mathrm{Li}_2(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{2}\left(\frac{1}{x}-x\right)\mathrm{Li}_2(1-x)-\frac{1}{4}(2+x)\ln(x)-\frac{5}{8x}+\frac{1}{2}+\frac{1}{8}x\,, \tag{A.39}$$

$$x^2\mathrm{Li}_2(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{3}\left(\frac{1}{x}-x^2\right)\mathrm{Li}_2(1-x)-\left(\frac{1}{3}+\frac{x}{6}+\frac{x^2}{9}\right)\ln(x)$$
$$-\frac{49}{108x}+\frac{1}{3}+\frac{x}{12}+\frac{x^2}{27}\,, \tag{A.40}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x^2\ln(x)$$
$$=x^2\left[\zeta_2-\mathrm{Li}_2(x)-\frac{1}{2}\ln^2(x)-\frac{3}{2}\ln(x)\right]-\frac{1}{4}-x+\frac{5}{4}x^2\,, \tag{A.41}$$

$$x^2\otimes\left(\frac{\ln^2(1-x)}{1-x}\right)_+$$

$$= \frac{x^2}{3}\ln^3(1-x) + \left(\frac{1}{2} + x - \frac{3}{2}x^2\right)\ln^2(1-x) - x(1-x)\ln(1-x) - x^2\ln(x) + 2x^2[\zeta_3 - \mathrm{S}_{1,2}(x)] + 3x^2[\zeta_2 - \mathrm{Li}_2(x)]\,, \tag{A.42}$$

$$x^2 \otimes \ln^2(1-x) = \frac{1}{2}(1-x^2)\ln^2(1-x) - x(1-x)\ln(1-x) + x^2[\zeta_2 - \mathrm{Li}_2(x) - \ln(x)]\,, \tag{A.43}$$

$$x^2 \otimes x\ln^2(1-x) = x(1-x)\ln^2(1-x) - 2x^2[\mathrm{Li}_2(x) - \zeta_2]\,, \tag{A.44}$$

$$x^2 \otimes \frac{\ln(x)\ln(1-x)}{1-x} = x^2[2\zeta_3 - \mathrm{Li}_3(x) - \mathrm{S}_{1,2}(x)] + \frac{3}{2}x^2\mathrm{Li}_2(1-x) + \frac{3}{4}x^2\ln^2(x) + \frac{1}{4}(1+4x-5x^2)\ln(1-x) + \ln(x)\left[\frac{x^2}{2}\ln^2(1-x) + x^2\mathrm{Li}_2(x) + \left(x+\frac{1}{2}\right)\ln(1-x) + \frac{5}{4}x^2 - \frac{x}{2}\right] - \frac{3}{4}x(1-x)\,, \tag{A.45}$$

$$x^2 \otimes \ln(x)\ln(1-x) = \frac{x^2}{2}\mathrm{Li}_2(1-x) + \frac{1}{4}(1-x^2)\ln(1-x) + \frac{x^2}{4}\ln^2(x) + \frac{1}{2}\left[\ln(1-x) - x + \frac{x^2}{2}\right]\ln(x) - \frac{3}{4}x(1-x)\,, \tag{A.46}$$

$$x^2 \otimes x\ln(x)\ln(1-x) = x(1-x)[1+\ln(x)]\ln(1-x) + x^2[\zeta_2 - \mathrm{Li}_2(x)] + x^2\ln(x)\left(1 + \frac{1}{2}\ln(x)\right), \tag{A.47}$$

$$x^2 \otimes \frac{\ln^2(x)}{1-x} = 2x^2\mathrm{S}_{1,2}(1-x) - \frac{x^2}{3}\ln^3(x) + \left(\frac{1}{2}+x\right)\ln^2(x) + \left(\frac{1}{2}+2x\right)\ln(x) + \frac{1}{4} + 2x - \frac{9}{4}x^2\,, \tag{A.48}$$

$$x^2 \otimes \ln^2(x) = \frac{1}{4}(1-x^2) + \frac{1}{2}[\ln(x) + \ln^2(x)]\,, \tag{A.49}$$

$$x^2 \otimes x\ln^2(x) = 2x(1-x) + x[\ln^2(x) + 2\ln(x)]\,, \tag{A.50}$$

$$x^2 \otimes \mathrm{Li}_2(1-x) = \frac{1}{2}(1-x^2)\mathrm{Li}_2(1-x) + \frac{1}{2}x(1-x) + \frac{x}{2}\left[\ln(x) - \frac{x}{2}\ln^2(x)\right], \tag{A.51}$$

$$x^2 \otimes x\mathrm{Li}_2(1-x) = x(1-x)\mathrm{Li}_2(1-x) - \frac{x^2}{2}\ln^2(x)\,, \tag{A.52}$$

$$\left(\frac{\ln^3(1-x)}{1-x}\right)_+ \otimes \frac{1}{x} = \frac{1}{4x}\ln^4(1-x)\,, \tag{A.53}$$

$$\frac{\ln(x)\ln^2(1-x)}{1-x} \otimes \frac{1}{x} = \frac{1}{x}\left\{\frac{1}{3}\ln(x)\ln^3(1-x) + 2[\mathrm{S}_{1,3}(x) - \zeta_4]\right\}, \tag{A.54}$$

$$\frac{\ln^2(x)\ln(1-x)}{1-x} \otimes \frac{1}{x} = \frac{1}{x}\left\{\frac{1}{2}\ln^2(x)\ln^2(1-x) + 2\left[\mathrm{S}_{2,2}(x) - \ln(x)\mathrm{S}_{1,2}(x) - \frac{\zeta_4}{4}\right]\right\}, \tag{A.55}$$

$$\frac{\ln^3(x)}{1-x} \otimes \frac{1}{x} = -\frac{6}{x}\mathrm{S}_{1,3}(1-x)\,, \tag{A.56}$$

$$\frac{\mathrm{S}_{1,2}(1-x)}{1-x} \otimes \frac{1}{x} = \frac{1}{x}\mathrm{S}_{2,2}(1-x)\,, \tag{A.57}$$

$$\frac{\mathrm{Li}_3(1-x)}{1-x} \otimes \frac{1}{x} = \frac{1}{x}\mathrm{Li}_4(1-x)\,, \tag{A.58}$$

$$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x} \otimes \frac{1}{x} = -\frac{1}{2x}\mathrm{Li}_2^2(1-x)\,, \tag{A.59}$$

$$\frac{\mathrm{Li}_3(x) - \zeta_3}{1-x} \otimes \frac{1}{x} = \frac{1}{x}\left\{\frac{1}{2}[\mathrm{Li}_2^2(x) - \zeta_2^2] + \ln(1-x)[\mathrm{Li}_3(x) - \zeta_3]\right\}, \tag{A.60}$$

$$\mathrm{S}_{1,2}(x) \otimes \frac{1}{x} = \frac{\zeta_3}{x} - \mathrm{S}_{1,2}(x) - \frac{1}{x}(1-x)\left[\frac{1}{2}\ln^2(1-x) - \ln(1-x) + 1\right], \tag{A.61}$$

$$x\mathrm{S}_{1,2}(x) \otimes \frac{1}{x} = \frac{1}{2x}\left[\zeta_3 - x^2\mathrm{S}_{1,2}(x)\right] - \frac{1}{8x}(1-x^2)\ln^2(1-x) + \frac{1}{8x}(3-2x-x^2)\ln(1-x) - \frac{1}{x}\left(\frac{7}{16} - \frac{3}{8}x - \frac{1}{16}x^2\right), \tag{A.62}$$

$$\mathrm{Li}_3(1-x) \otimes \frac{1}{x} = \left(\frac{1}{x} - 1\right)[\mathrm{Li}_3(1-x) - \mathrm{Li}_2(1-x) + 1] + \ln(x)\,, \tag{A.63}$$

$$x\mathrm{Li}_3(1-x) \otimes \frac{1}{x} = \frac{1}{2x}(1-x^2)\mathrm{Li}_3(1-x) - \left(\frac{3}{4x} - \frac{1}{2} - \frac{1}{4}x\right)\mathrm{Li}_2(1-x) + \left(\frac{3}{4} + \frac{x}{8}\right)\ln(x) + \frac{13}{16x} - \frac{3}{4} - \frac{x}{16}\,, \tag{A.64}$$

$$\ln(1-x)\mathrm{Li}_2(1-x) \otimes \frac{1}{x} = -\frac{1}{x}(1-x)[1-\ln(1-x)]\mathrm{Li}_2(1-x) - \frac{1}{x}(1-x)[1-\ln(x)]\ln(1-x) + 2\ln(x) + \frac{1}{x}[\mathrm{Li}_2(x) - \zeta_2] + \frac{3}{x}(1-x)\,, \tag{A.65}$$

$$x\ln(1-x)\mathrm{Li}_2(1-x) \otimes \frac{1}{x} = \frac{1-x}{x}\left\{[\ln(1-x) - 1] - \frac{1-x}{2}\left[\ln(1-x) - \frac{1}{2}\right]\right\}\mathrm{Li}_2(1-x) + \frac{1}{x}\left\{\left[\frac{3}{4} - \frac{x}{2} - \frac{x^2}{4}\right]\ln(1-x) + \frac{3}{2}x + \frac{x^2}{4}\right\}\ln(x)$$

$$-\frac{1}{8x}[5-4x-x^2]\ln(1-x)+\frac{3}{4x}[\mathrm{Li}_2(x)-\zeta_2] + \frac{37}{16x}-\frac{17}{8}-\frac{3}{16}x\,, \quad (A.66)$$

$$\ln(x)\mathrm{Li}_2(1-x)\otimes\frac{1}{x} = -\frac{1}{x}\{2\mathrm{S}_{1,2}(1-x)+x\ln^2(x) + [1-x+x\ln(x)][\mathrm{Li}_2(1-x)-3]\}\,, \quad (A.67)$$

$$x\ln(x)\mathrm{Li}_2(1-x)\otimes\frac{1}{x} = -\frac{1}{x}\mathrm{S}_{1,2}(1-x)-\frac{1}{4x}[1-x^2+2x^2\ln(x)]\mathrm{Li}_2(1-x) -\frac{1}{4}(2+x)\ln^2(x)+\frac{1}{8}(10+3x)\ln(x) +\left(\frac{23}{16x}-\frac{5}{4}-\frac{3}{16}x\right), \quad (A.68)$$

$$\mathrm{S}_{1,2}(1-x)\otimes\frac{1}{x} = \left(\frac{1}{x}-1\right)[\mathrm{S}_{1,2}(1-x)-1]+\frac{1}{2}\ln^2(x)-\ln(x)\,, \quad (A.69)$$

$$x\mathrm{S}_{1,2}(1-x)\otimes\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x}-x\right)\mathrm{S}_{1,2}(1-x)+\frac{1}{8}(2+x)\ln^2(x) -\frac{1}{8}(4+x)\ln(x)-\frac{9}{16x}+\frac{1}{2}+\frac{x}{16}\,, \quad (A.70)$$

$$\mathrm{Li}_3(x)\otimes\frac{1}{x} = -\left(\frac{1}{x}-1\right)[\ln(1-x)-1]-\frac{1}{x}(\zeta_2-\zeta_3)+\mathrm{Li}_2(x)-\mathrm{Li}_3(x)\,, \quad (A.71)$$

$$x\mathrm{Li}_3(x)\otimes\frac{1}{x} = \frac{1}{x}\left(\frac{3}{16}-\frac{1}{4}\zeta_2+\frac{1}{2}\zeta_3\right)-\frac{1}{8}\left(1+\frac{x}{2}\right) -\frac{1}{8}\left(\frac{1}{x}-x\right)\ln(1-x)+x\left[\frac{1}{4}\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_3(x)\right], \quad (A.72)$$

$$\ln(x)\mathrm{Li}_2(x)\otimes\frac{1}{x} = \frac{1}{x}\{x[1-\ln(x)]\mathrm{Li}_2(x)-\zeta_2-\mathrm{Li}_2(1-x) +3(1-x)-[2(1-x)+x\ln(x)]\ln(1-x)+x\ln(x)\}\,, \quad (A.73)$$

$$x\ln(x)\mathrm{Li}_2(x)\otimes\frac{1}{x} = \frac{1}{4x}\{x^2[1-2\ln(x)]\mathrm{Li}_2(x)-\zeta_2\} -\frac{1}{4x}[1-x^2+x^2\ln(x)]\ln(1-x) +\frac{11}{16x}-\frac{1}{2}-\frac{3}{16}x+\frac{1}{8}(x+2)\ln(x)-\frac{1}{4x}\mathrm{Li}_2(1-x)\,, \quad (A.74)$$

$$\ln^3(1-x)\otimes\frac{1}{x} = \left(\frac{1}{x}-1\right)[\ln^3(1-x)-3\ln^2(1-x)+6\ln(1-x)-6]\,, \quad (A.75)$$

$$x\ln^3(1-x)\otimes\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x}-x\right)\ln^3(1-x)+\left(-\frac{9}{4x}+\frac{3}{2}+\frac{3}{4}x\right)\ln^2(1-x) +\left(\frac{21}{4x}-\frac{9}{2}-\frac{3}{4}x\right)\ln(1-x)-\frac{45}{8x}+\frac{21}{4}+\frac{3}{8}x\,, \quad (A.76)$$

$$\ln(x)\ln^2(1-x)\otimes\frac{1}{x} = \frac{1}{x}(1-x)[\ln(x)-1]\ln^2(1-x) +\frac{4}{x}\left[1-x+\frac{x}{2}\ln(x)\right]\ln(1-x)-2\ln(x) -\frac{2}{x}[\mathrm{S}_{1,2}(x)-\zeta_3]+\frac{2}{x}\mathrm{Li}_2(1-x)-\frac{6}{x}(1-x)\,, \quad (A.77)$$

$$x\ln(x)\ln^2(1-x)\otimes\frac{1}{x} = \frac{1}{x}\left\{\zeta_3-\mathrm{S}_{1,2}(x)-\frac{3}{2}[\mathrm{Li}_2(x)-\zeta_2]\right\} -\frac{1}{4x}(1-x^2)[1-2\ln(x)]\ln^2(1-x) +\left\{\frac{2}{x}-\frac{3}{2}-\frac{x}{2}+\frac{1}{x}\left[-\frac{3}{2}+x+\frac{x^2}{2}\right]\ln(x)\right\}\ln(1-x) -\frac{1}{4}(x+6)\ln(x)-\frac{31}{8x}+\frac{7}{2}+\frac{3}{8}x\,, \quad (A.78)$$

$$\ln^2(x)\ln(1-x)\otimes\frac{1}{x} = \frac{2}{x}[\mathrm{S}_{1,2}(1-x)+\mathrm{Li}_2(1-x)+\ln(1-x)-3(1-x)] +\ln(x)[\ln(x)-4]-[\ln^2(x)-2\ln(x)+2]\ln(1-x)\,, \quad (A.79)$$

$$x\ln^2(x)\ln(1-x)\otimes\frac{1}{x} = \frac{1}{x}\left[\mathrm{S}_{1,2}(1-x)+\frac{1}{2}\mathrm{Li}_2(1-x)\right] +\frac{1}{4x}(1-x^2)\ln(1-x)+\frac{1}{2}\left[1+\frac{x}{2}-x\ln(1-x)\right]\ln^2(x) -\frac{1}{2}[3+x-x\ln(1-x)]\ln(x)-\frac{17}{8x}+\frac{7}{4}+\frac{3}{8}x\,, \quad (A.80)$$

$$\ln^3(x)\otimes\frac{1}{x} = -6\left(\frac{1}{x}-1\right)-\ln^3(x)+3\ln^2(x)-6\ln(x)\,, \quad (A.81)$$

$$x\ln^3(x)\otimes\frac{1}{x} = -\frac{3}{8}\left(\frac{1}{x}-x\right)-\frac{x}{2}\left[\ln^3(x)-\frac{3}{2}\ln^2(x)+\frac{3}{2}\ln(x)\right], \quad (A.82)$$

$$\frac{1}{x}\mathrm{Li}_2(1-x)\otimes\frac{1}{x} = -\frac{1}{x}[2\mathrm{S}_{1,2}(1-x)+\ln(x)\mathrm{Li}_2(1-x)]\,, \quad (A.83)$$

$$\frac{1}{x}\mathrm{Li}_2(1-x)\otimes 1 = \left(\frac{1}{x}-1\right)\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2(x)\,, \qquad (A.84)$$

$$\frac{1}{x}\mathrm{Li}_2(1-x)\otimes x = \frac{1}{2}(1-x) + \frac{1}{2}\left(\frac{1}{x}-x\right)\mathrm{Li}_2(1-x) + \frac{1}{2}\ln(x)\left[1-\frac{1}{2}x\ln(x)\right]\,, \qquad (A.85)$$

$$\frac{1}{x}\mathrm{Li}_2(1-x)\otimes x^2 = \frac{1}{3}\left(\frac{1}{x}-x^2\right)\mathrm{Li}_2(1-x) + \frac{1}{6}\ln(x)[1+2x-x^2\ln(x)] + \frac{1}{12}[1+4x-5x^2]\,, \qquad (A.86)$$

$$1\otimes x^2\ln^2(1-x) = \frac{1}{2}(1-x^2)\ln^2(1-x) + \frac{1}{2}(-3+2x+x^2)\ln(1-x) + \frac{7}{4} - \frac{3}{2}x - \frac{x^2}{4}\,, \qquad (A.87)$$

$$1\otimes x^2\ln(x)\ln(1-x) = \frac{1}{2}[\mathrm{Li}_2(x)-\zeta_2] + \frac{1}{2}(1-x^2)\ln(x)\ln(1-x) - \frac{1}{4}(1-x^2)\ln(1-x) + \frac{x}{4}(2+x)\ln(x) + 1 - \frac{3}{4}x - \frac{x^2}{4}\,, \qquad (A.88)$$

$$1\otimes x^2\ln^2(x) = \frac{x^2}{2}[\ln(x)-\ln^2(x)] + \frac{1}{4}(1-x^2)\,, \qquad (A.89)$$

$$1\otimes \frac{1}{x}\ln(1-x) = \left(\frac{1}{x}-1\right)\ln(1-x) + \ln(x)\,, \qquad (A.90)$$

$$1\otimes x^2\mathrm{Li}_2(1-x) = \frac{1}{2}(1-x^2)\mathrm{Li}_2(1-x) - \frac{x}{4}(2+x)\ln(x) - \frac{5}{8} + \frac{x}{2} + \frac{x^2}{8}\,, \qquad (A.91)$$

$$x\otimes x^2\ln^2(1-x) = x(1-x)[\ln^2(1-x) - 2\ln(1-x) + 2]\,, \qquad (A.92)$$

$$x\otimes x^2\ln(x)\ln(1-x) = -x\mathrm{Li}_2(1-x) + x(1-x)[2-\ln(1-x)] + x^2\ln(x)[1-\ln(1-x)]\,, \qquad (A.93)$$

$$x\otimes x^2\ln^2(x) = -x^2\ln^2(x) + 2x[x\ln(x) - x + 1]\,, \qquad (A.94)$$

$$x\otimes \frac{1}{x}\ln(1-x) = \frac{1}{2}\left(\frac{1}{x}-x\right)\ln(1-x) + \frac{x}{2}\ln(x) - \frac{1}{2}(1-x)\,, \qquad (A.95)$$

$$x^2\otimes \frac{1}{(1-x)_+} = x^2[\ln(1-x)-\ln(x)] + \frac{1}{2} + x - \frac{3}{2}x^2\,, \qquad (A.96)$$

$$x^2\otimes \ln(1-x) = \frac{1}{2}(1-x^2)\ln(1-x) + \frac{x}{2}[x\ln(x) + x - 1]\,, \qquad (A.97)$$

$$x^2\otimes x\ln(1-x) = x(1-x)\ln(1-x) + x^2\ln(x)\,, \qquad (A.98)$$

$$x^2\otimes\left(\frac{\ln(1-x)}{1-x}\right)_+ = -x^2\mathrm{Li}_2(1-x) + \frac{x^2}{2}\ln^2(1-x) - x^2\ln(x)\ln(1-x) + \frac{3}{2}x^2\ln(x) + \left(\frac{1}{2}+x-\frac{3}{2}x^2\right)\ln(1-x) - \frac{1}{2}x(1-x)\,, \qquad (A.99)$$

$$x^2\otimes\frac{\ln(x)}{1-x} = -x^2\mathrm{Li}_2(1-x) - \frac{x^2}{2}\ln^2(x) + \left(x+\frac{1}{2}\right)\ln(x) + \frac{1}{4}(1+4x-5x^2)\,, \qquad (A.100)$$

$$x^2\otimes\frac{1}{x}\ln(1-x) = \frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln(1-x) + \frac{x^2}{3}\ln(x) - \frac{1}{6} - \frac{x}{3} + \frac{x^2}{2}\,, \qquad (A.101)$$

$$x^2\otimes x^2\ln(1-x) = x^2[\mathrm{Li}_2(x)-\zeta_2]\,, \qquad (A.102)$$

$$x^2\otimes x^2\ln(x) = -\frac{x^2}{2}\ln^2(x)\,, \qquad (A.103)$$

$$x\otimes x^2\mathrm{Li}_2(1-x) = x(1-x)[\mathrm{Li}_2(1-x)-1] - x^2\ln(x)\,, \qquad (A.104)$$

$$x^2\otimes\left(\frac{\ln^3(1-x)}{1-x}\right)_+ = 3x^2\{2[\mathrm{S}_{1,3}(x)-\zeta_4] + 3[\mathrm{S}_{1,2}(x)-\zeta_3] + [\mathrm{Li}_2(x)-\zeta_2]\} + \frac{x^2}{4}\ln^4(1-x) + \frac{1}{2}(1+2x-3x^2)\ln^3(1-x) - \frac{3}{2}x(1-x)\ln^2(1-x)\,, \qquad (A.105)$$

$$\begin{aligned} x^2\otimes\frac{\ln(x)\ln^2(1-x)}{1-x} &= 2x^2\left[\mathrm{S}_{2,2}(x) + \mathrm{S}_{1,3}(x) - \ln(x)\mathrm{S}_{1,2}(x) - \frac{5}{4}\zeta_4\right] \\ &+ 3x^2[\mathrm{S}_{1,2}(x) - \mathrm{S}_{1,2}(1-x) - \zeta_3] - 2x^2[\mathrm{Li}_2(x)-\zeta_2] \\ &- \frac{x^2}{2}\mathrm{Li}_2(1-x) + \frac{x^2}{3}\ln(x)\ln^3(1-x) - \frac{1}{2}x^2[\ln^2(x) + 3\ln(x)] \\ &+ \left[\frac{1}{4} + x - \frac{5}{4}x^2 + \left(\frac{1}{2}+x-\frac{3}{2}x^2\right)\ln(x)\right]\ln^2(1-x) \\ &+ \left[-\frac{3}{2}x(1-x) + \left(\frac{x^2}{2}-x\right)\ln(x) + \frac{3}{2}x^2\ln^2(x)\right] \times\ln(1-x)\,, \end{aligned} \qquad (A.106)$$

$$\begin{aligned} x^2\otimes\frac{\ln^2(x)\ln(1-x)}{1-x} &= 2x^2\left[\left\{\ln(1-x)-\frac{3}{2}\right\}\mathrm{S}_{1,2}(1-x) - \mathrm{S}_{2,2}(1-x) - \mathrm{S}_{1,3}(1-x)\right] \\ &+ \frac{5}{2}x^2\mathrm{Li}_2(1-x) + x^2\left(\frac{1}{2}-\frac{1}{3}\ln(1-x)\right)\ln^3(x) \\ &+ \left[-\frac{x}{2} + \frac{5}{4}x^2 + \left(\frac{1}{2}+x\right)\ln(1-x)\right]\ln^2(x) \end{aligned}$$

$$+\left[-\frac{3}{2}x+\frac{9}{4}x^2+\left(\frac{1}{2}+2x\right)\ln(1-x)\right]\ln(x)$$
$$+\left(\frac{1}{4}+2x-\frac{9}{4}x^2\right)\ln(1-x)-\frac{7}{4}x(1-x)\,, \qquad \text{(A.107)}$$

$$x^2\otimes\frac{\ln^3(x)}{1-x}$$
$$=-6x^2\mathrm{S}_{1,3}(1-x)-\frac{x^2}{4}\ln^4(x)+\left(x+\frac{1}{2}\right)\ln^3(x)$$
$$+3\left(x+\frac{1}{4}\right)\ln^2(x)+6\left(x+\frac{1}{8}\right)\ln(x)+\frac{3}{8}+6x-\frac{51}{8}x^2\,, \qquad \text{(A.108)}$$

$$x^2\otimes\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}$$
$$=3x^2\left[\mathrm{S}_{1,3}(1-x)+\mathrm{S}_{1,2}(1-x)-\frac{1}{6}\mathrm{Li}_2^2(1-x)\right]$$
$$+\frac{1}{2}\left[-x^2\ln^2(x)+(1+2x)\ln(x)+\frac{1}{2}+2x-\frac{5}{2}x^2\right]$$
$$\times\mathrm{Li}_2(1-x)$$
$$-\frac{x^2}{2}\ln^3(x)+\frac{x}{2}\left(1-\frac{5}{4}x\right)\ln^2(x)+\frac{5}{4}x\ln(x)+\frac{5}{4}x(1-x)\,, \qquad \text{(A.109)}$$

$$x^2\otimes\mathrm{Li}_3(1-x)$$
$$=\frac{1}{2}(1-x^2)\mathrm{Li}_3(1-x)-\frac{1}{2}[x(1-x)-x^2\ln(x)]\mathrm{Li}_2(1-x)$$
$$+\frac{x^2}{4}\ln^2(x)+x^2\mathrm{S}_{1,2}(1-x)\,, \qquad \text{(A.110)}$$

$$x^2\otimes x\mathrm{Li}_3(1-x)$$
$$=x(1-x)\mathrm{Li}_3(1-x)+x^2\ln(x)\mathrm{Li}_2(1-x)+2x^2\mathrm{S}_{1,2}(1-x)\,, \qquad \text{(A.111)}$$

$$x^2\otimes\ln(1-x)\mathrm{Li}_2(1-x)$$
$$=\frac{3}{2}x^2\mathrm{S}_{1,2}(1-x)$$
$$+\frac{1}{2}[(1-x^2)\ln(1-x)+x^2\ln(x)-x+2x^2]\mathrm{Li}_2(1-x)$$
$$+\frac{x}{2}\left[-\frac{x}{2}\ln^2(x)+\ln(x)+1-x\right]\ln(1-x)$$
$$+\frac{x^2}{2}[\ln^2(x)+\ln(x)]\,, \qquad \text{(A.112)}$$

$$x^2\otimes x\ln(1-x)\mathrm{Li}_2(1-x)$$
$$=x^2[2\mathrm{S}_{1,2}(1-x)+\ln(x)\mathrm{Li}_2(x)-\mathrm{Li}_3(x)+\zeta_3]$$
$$+x[x\ln(x)+(1-x)\ln(1-x)]\mathrm{Li}_2(1-x)\,, \qquad \text{(A.113)}$$

$$x^2\otimes\ln(x)\mathrm{Li}_2(1-x)$$
$$=x^2\mathrm{S}_{1,2}(1-x)+\frac{1}{2}\left[\ln(x)+\frac{1}{2}(1-x^2)\right]\mathrm{Li}_2(1-x)$$
$$-\frac{x^2}{6}\ln^3(x)+\frac{x}{8}(4-x)\ln^2(x)+\frac{5}{4}x[1-x+\ln(x)]\,, \qquad \text{(A.114)}$$

$$x^2\otimes x\ln(x)\mathrm{Li}_2(1-x)$$
$$=2x^2\mathrm{S}_{1,2}(1-x)+x[1-x+\ln(x)]\mathrm{Li}_2(1-x)$$
$$-x^2\left[\frac{1}{3}\ln^3(x)+\frac{1}{2}\ln^2(x)\right]\,, \qquad \text{(A.115)}$$

$$x^2\otimes\mathrm{S}_{1,2}(1-x)$$
$$=\frac{1}{2}(1-x^2)\mathrm{S}_{1,2}(1-x)+\frac{x^2}{12}\ln^3(x)-\frac{x}{4}\ln^2(x)-\frac{x}{2}\ln(x)$$
$$-\frac{x}{2}(1-x)\,, \qquad \text{(A.116)}$$

$$x^2\otimes x\mathrm{S}_{1,2}(1-x)=x(1-x)\mathrm{S}_{1,2}(1-x)+\frac{x^2}{6}\ln^3(x)\,, \qquad \text{(A.117)}$$

$$x^2\otimes\ln^3(1-x)$$
$$=\frac{1}{2}(1-x^2)\ln^3(1-x)-\frac{3}{2}x(1-x)\ln^2(1-x)$$
$$+3x^2[\mathrm{S}_{1,2}(x)-\zeta_3+\mathrm{Li}_2(x)-\zeta_2]\,, \qquad \text{(A.118)}$$

$$x^2\otimes x\ln^3(1-x)=x(1-x)\ln^3(1-x)+6x^2[\mathrm{S}_{1,2}(x)-\zeta_3]\,, \qquad \text{(A.119)}$$

$$x^2\otimes\ln(x)\ln^2(1-x)$$
$$=x^2[\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(1-x)-\zeta_3]-\frac{x^2}{2}\mathrm{Li}_2(1-x)$$
$$+\frac{1}{2}\left[\frac{1}{2}+\ln(x)\right](1-x^2)\ln^2(1-x)$$
$$+\left[-\frac{3}{2}x(1-x)-x\left(1-\frac{x}{2}\right)\ln(x)+\frac{x^2}{2}\ln^2(x)\right]\ln(1-x)$$
$$-\frac{x^2}{2}[3\ln(x)+\ln^2(x)]\,, \qquad \text{(A.120)}$$

$$x^2\otimes x\ln(x)\ln^2(1-x)$$
$$=x(1-x)[1+\ln(x)]\ln^2(1-x)+2x^2\ln(x)\ln(1-x)$$
$$+2x^2[\mathrm{S}_{1,2}(x)+\mathrm{Li}_3(x)+\mathrm{Li}_2(1-x)-\ln(x)\mathrm{Li}_2(x)-2\zeta_3]\,, \qquad \text{(A.121)}$$

$$x^2\otimes\ln^2(x)\ln(1-x)$$
$$=-x^2\mathrm{S}_{1,2}(1-x)+\frac{x^2}{2}\mathrm{Li}_2(1-x)$$
$$+\left\{\frac{1}{4}(1-x^2)+\frac{1}{2}\ln(x)[1+\ln(x)]\right\}\ln(1-x)$$
$$+\frac{x^2}{6}\ln^3(x)-\frac{x}{2}\left(1-\frac{x}{2}\right)\ln^2(x)-\frac{x}{2}\left(3-\frac{x}{2}\right)\ln(x)$$
$$-\frac{7}{4}x(1-x)\,, \qquad \text{(A.122)}$$

$$x^2\otimes x\ln^2(x)\ln(1-x)$$
$$=x[2(1-x)+\ln^2(x)+2\ln(x)]\ln(1-x)$$
$$+2x^2\left[\frac{1}{6}\ln^3(x)+\frac{1}{2}\ln^2(x)+\ln(x)\right]$$
$$+2x^2[\mathrm{Li}_2(1-x)-\mathrm{S}_{1,2}(1-x)]\,, \qquad \text{(A.123)}$$

$$x^2\otimes\ln^3(x)=\frac{1}{2}\ln^3(x)+\frac{3}{4}\ln^2(x)+\frac{3}{4}\ln(x)+\frac{3}{8}(1-x^2)\,, \qquad \text{(A.124)}$$

$$x^2\otimes x\ln^3(x)=x[\ln^3(x)+3\ln^2(x)+6\ln(x)]+6x(1-x)\,, \qquad \text{(A.125)}$$

$$x^2\mathrm{S}_{1,2}(1-x)\otimes\frac{1}{x}$$
$$=\frac{1}{3}\left(\frac{1}{x}-x^2\right)\mathrm{S}_{1,2}(1-x)+\frac{1}{6}\left[1+\frac{x}{2}+\frac{x^2}{3}\right]\ln^2(x)$$

$$-\frac{1}{3}\left[1+\frac{x}{4}+\frac{x^2}{9}\right]\ln(x)-\frac{251}{648x}+\frac{1}{3}+\frac{x}{24}+\frac{x^2}{81}, \tag{A.126}$$

$$x^2\mathrm{Li}_3(1-x)\otimes\frac{1}{x} = \frac{1}{3}\left(\frac{1}{x}-x^2\right)\mathrm{Li}_3(1-x) + \left[-\frac{11}{18x}+\frac{1}{3}+\frac{x}{6}+\frac{x^2}{9}\right]\mathrm{Li}_2(1-x) + \left[\frac{11}{18}+\frac{5}{36}x+\frac{x^2}{27}\right]\ln(x)+\frac{449}{648x}-\frac{11}{18}-\frac{5}{72}x-\frac{x^2}{81}, \tag{A.127}$$

$$\frac{1}{x}\ln^2(1-x)\otimes\frac{1}{x}=\frac{2}{x}[\zeta_3-\mathrm{S}_{1,2}(x)], \tag{A.128}$$

$$\frac{1}{x}\ln(x)\otimes\frac{1}{x}=-\frac{1}{2x}\ln^2(x), \tag{A.129}$$

$$1\otimes x^2\mathrm{S}_{1,2}(1-x) = \frac{1}{2}(1-x^2)\mathrm{S}_{1,2}(1-x)+\frac{x}{4}\left(1+\frac{x}{2}\right)\ln^2(x) -\frac{x}{2}\left(1+\frac{x}{4}\right)\ln(x)-\frac{9}{16}+\frac{x}{2}+\frac{x^2}{16}, \tag{A.130}$$

$$x\otimes x^2\mathrm{S}_{1,2}(1-x) = x(1-x)[\mathrm{S}_{1,2}(1-x)-1]+x^2\ln(x)\left[\frac{1}{2}\ln(x)-1\right], \tag{A.131}$$

$$1\otimes x^2\mathrm{Li}_3(1-x) = \frac{1}{2}(1-x^2)\mathrm{Li}_3(1-x)-\frac{1}{4}(3-2x-x^2)\mathrm{Li}_2(1-x) +\frac{x}{4}\left(3+\frac{x}{2}\right)\ln(x)+\frac{13}{16}-\frac{3}{4}x-\frac{x^2}{16}, \tag{A.132}$$

$$x\otimes x^2\mathrm{Li}_3(1-x) = x(1-x)[\mathrm{Li}_3(1-x)-\mathrm{Li}_2(1-x)+1]+x^2\ln(x), \tag{A.133}$$

$$x^2\otimes x^2\mathrm{Li}_3(1-x)=-x^2\ln(x)\mathrm{Li}_3(1-x)-\frac{x^2}{2}\mathrm{Li}_2^2(1-x), \tag{A.134}$$

$$1\otimes\frac{1}{x}\ln^2(1-x)=\left(\frac{1}{x}-1\right)\ln^2(1-x)-2[\mathrm{Li}_2(x)-\zeta_2], \tag{A.135}$$

$$x\otimes\frac{1}{x}\ln^2(1-x) = \frac{1}{2}\left(\frac{1}{x}-x\right)\ln^2(1-x)-(1-x)\ln(1-x)-x\ln(x) -x[\mathrm{Li}_2(x)-\zeta_2], \tag{A.136}$$

$$x^2\otimes\frac{1}{x}\ln^2(1-x) = \frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln^2(1-x)-\frac{1}{3}(1+2x-3x^2)\ln(1-x) -x^2\ln(x)-\frac{2}{3}x^2[\mathrm{Li}_2(x)-\zeta_2]+\frac{1}{3}x(1-x), \tag{A.137}$$

$$1\otimes\frac{1}{x}\ln(x)=\frac{1}{x}[1-x+\ln(x)], \tag{A.138}$$

$$x\otimes\frac{1}{x}\ln(x)=\frac{1}{2x}\left[\frac{1}{2}(1-x^2)+\ln(x)\right], \tag{A.139}$$

$$x^2\otimes\frac{1}{x}\ln(x)=\frac{1}{3x}\left[\frac{1}{3}(1-x^3)+\ln(x)\right], \tag{A.140}$$

$$x^2\ln(1-x)\mathrm{Li}_2(1-x)\otimes\frac{1}{x} = \left\{-\frac{11}{18x}+\frac{1}{3}+\frac{x}{6}+\frac{x^2}{9}+\frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln(1-x)\right\} \times\mathrm{Li}_2(1-x) +\frac{11}{18x}[\mathrm{Li}_2(x)-\zeta_2] +\left(\frac{11}{18x}-\frac{1}{3}-\frac{x}{6}-\frac{x^2}{9}\right)\ln(x)\ln(1-x) +\left(-\frac{49}{108x}+\frac{1}{3}+\frac{x}{12}+\frac{x^2}{27}\right)\ln(1-x) +\left(\frac{11}{9}+\frac{5}{18}x+\frac{2}{27}x^2\right)\ln(x)+\frac{413}{216x}-\frac{181}{108}-\frac{43}{216}x-\frac{x^2}{27}, \tag{A.141}$$

$$x^2\ln^3(1-x)\otimes\frac{1}{x} = \frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln^3(1-x) +\left(-\frac{11}{6x}+1+\frac{x}{2}+\frac{x^2}{3}\right)\ln^2(1-x) +\left(\frac{85}{18x}-\frac{11}{3}-\frac{5}{6}x-\frac{2}{9}x^2\right)\ln(1-x) -\frac{575}{108x}+\frac{85}{18}+\frac{19}{36}x+\frac{2}{27}x^2, \tag{A.142}$$

$$x^2\ln(x)\ln^2(1-x)\otimes\frac{1}{x} = \frac{2}{3x}[\zeta_3-\mathrm{S}_{1,2}(x)]-\frac{11}{9x}[\mathrm{Li}_2(x)-\zeta_2] +\frac{1}{3}\left(\frac{1}{x}-x^2\right)\ln(x)\ln^2(1-x)-\frac{1}{9}\left(\frac{1}{x}-x^2\right)\ln^2(1-x) +\left(-\frac{11}{9x}+\frac{2}{3}+\frac{x}{3}+\frac{2}{9}x^2\right)\ln(x)\ln(1-x) +\left(\frac{71}{54x}-\frac{8}{9}-\frac{5}{18}x-\frac{4}{27}x^2\right)\ln(1-x) -\left(\frac{11}{9}+\frac{5}{18}x+\frac{2}{27}x^2\right)\ln(x)-\frac{80}{27x}+\frac{137}{54}+\frac{19}{54}x+\frac{2}{27}x^2, \tag{A.143}$$

$$x^2\ln^2(x)\ln(1-x)\otimes\frac{1}{x} = \frac{2}{3x}\mathrm{S}_{1,2}(1-x)+\frac{2}{9x}\mathrm{Li}_2(1-x) +\left(\frac{1}{3}+\frac{x}{6}+\frac{x^2}{9}-\frac{x^2}{3}\ln(1-x)\right)\ln^2(x) +\frac{2}{9}x^2\ln(x)\ln(1-x)-\left(\frac{8}{9}+\frac{5}{18}x+\frac{4}{27}x^2\right)\ln(x) +\frac{2}{27}\left(\frac{1}{x}-x^2\right)\ln(1-x)-\frac{131}{108x}+\frac{26}{27}+\frac{19}{108}x+\frac{2}{27}x^2, \tag{A.144}$$

$$x^2 \ln^3(x) \otimes \frac{1}{x} = -\frac{2}{27}\left(\frac{1}{x} - x^2\right) - \frac{x^2}{3}\left(\frac{2}{3}\ln(x) - \ln^2(x) + \ln^3(x)\right), \tag{A.145}$$

$$\begin{aligned} 1 \otimes x^2 \ln(1-x)\mathrm{Li}_2(1-x) &= \frac{1-x}{2}\left\{(1+x)\ln(1-x) - \frac{3}{2} - \frac{x}{2}\right\}\mathrm{Li}_2(1-x) \\ &+ \frac{3}{4}[\mathrm{Li}_2(x) - \zeta_2] + \left(\frac{3}{4} - \frac{x}{2} - \frac{x^2}{4}\right)\ln(x)\ln(1-x) \\ &+ \left(\frac{3}{2} + \frac{x}{4}\right)x\ln(x) - \frac{1}{8}(5 - 4x - x^2)\ln(1-x) \\ &+ \frac{37}{16} - \frac{17}{8}x - \frac{3}{16}x^2, \end{aligned} \tag{A.146}$$

$$\begin{aligned} 1 \otimes x^2 \ln^3(1-x) &= \frac{1}{2}(1-x^2)\ln^3(1-x) + \left(-\frac{9}{4} + \frac{3}{2}x + \frac{3}{4}x^2\right)\ln^2(1-x) \\ &+ \left(\frac{21}{4} - \frac{9}{2}x - \frac{3}{4}x^2\right)\ln(1-x) - \frac{45}{8} + \frac{21}{4}x + \frac{3}{8}x^2, \end{aligned} \tag{A.147}$$

$$\begin{aligned} 1 \otimes x^2 \ln(x)\ln^2(1-x) &= \zeta_3 - \mathrm{S}_{1,2}(x) - \frac{3}{2}[\mathrm{Li}_2(x) - \zeta_2] \\ &- \frac{1}{4}(1-x^2)[1 - 2\ln(x)]\ln^2(1-x) \\ &+ \left(-\frac{3}{2} + x + \frac{x^2}{2}\right)\ln(x)\ln(1-x) \\ &+ \left(2 - \frac{3}{2}x - \frac{x^2}{2}\right)\ln(1-x) \\ &- \frac{x}{4}(x+6)\ln(x) - \frac{31}{8} + \frac{7}{2}x + \frac{3}{8}x^2, \end{aligned} \tag{A.148}$$

$$\begin{aligned} 1 \otimes x^2 \ln^2(x)\ln(1-x) &= \mathrm{S}_{1,2}(1-x) + \frac{1}{2}\mathrm{Li}_2(1-x) \\ &+ \frac{x}{2}\left[1 + \frac{x}{2} - x\ln(1-x)\right]\ln^2(x) \\ &- \frac{x}{2}[3 + x - x\ln(1-x)]\ln(x) \\ &+ \frac{1}{4}(1-x^2)\ln(1-x) - \frac{17}{8} + \frac{7}{4}x + \frac{3}{8}x^2, \end{aligned} \tag{A.149}$$

$$1 \otimes x^2 \ln^3(x) = -\frac{x^2}{2}\left\{\ln^3(x) - \frac{3}{2}\ln^2(x) + \frac{3}{2}\ln(x)\right\} - \frac{3}{8}(1-x^2), \tag{A.150}$$

$$\begin{aligned} x \otimes x^2 \ln(1-x)\mathrm{Li}_2(1-x) &= -x\mathrm{Li}_2(1-x) + x(1-x)[\ln(1-x) - 1]\mathrm{Li}_2(1-x) \\ &+ x(1-x)[3 - \ln(1-x)] + x^2\ln(x)[2 - \ln(1-x)], \end{aligned} \tag{A.151}$$

$$x \otimes x^2 \ln^3(1-x) = -x(1-x)[6 - 6\ln(1-x) + 3\ln^2(1-x) - \ln^3(1-x)], \tag{A.152}$$

$$\begin{aligned} x \otimes x^2 \ln(x)\ln^2(1-x) &= 2x[\zeta_3 - \mathrm{S}_{1,2}(x)] + 2x[\zeta_2 - \mathrm{Li}_2(x)] - 2x\ln(x) \\ &- x(1-x)[6 - 4\ln(1-x) + \ln^2(1-x)] \\ &+ x(1-x)\ln(x)[2 - 2\ln(1-x) + \ln^2(1-x)], \end{aligned} \tag{A.153}$$

$$\begin{aligned} x \otimes x^2 \ln^2(x)\ln(1-x) &= 2x[\mathrm{S}_{1,2}(1-x) + \mathrm{Li}_2(1-x)] \\ &+ x^2\ln(x)\{-2 + [2 - \ln(x)][\ln(1-x) - 1]\} \\ &+ 2x(1-x)[\ln(1-x) - 3], \end{aligned} \tag{A.154}$$

$$x \otimes x^2 \ln^3(x) = -x^2\ln^3(x) + 3x^2\ln^2(x) - 6x(x\ln(x) - x + 1), \tag{A.155}$$

$$x^2 \otimes x^2 \ln^2(1-x) = 2x^2[\zeta_3 - \mathrm{S}_{1,2}(x)], \tag{A.156}$$

$$x^2 \otimes x^2 \ln(x)\ln(1-x) = x^2[\zeta_3 + \ln(x)\mathrm{Li}_2(x) - \mathrm{Li}_3(x)], \tag{A.157}$$

$$x^2 \otimes x^2 \ln^2(x) = -\frac{x^2}{3}\ln^3(x), \tag{A.158}$$

$$x^2 \otimes x^2\mathrm{Li}_2(1-x) = -x^2[2\mathrm{S}_{1,2}(1-x) + \ln(x)\mathrm{Li}_2(1-x)], \tag{A.159}$$

$$\begin{aligned} x^2 \otimes \frac{\mathrm{S}_{1,2}(1-x)}{1-x} &= x^2[\mathrm{S}_{2,2}(1-x) - 3\mathrm{S}_{1,3}(1-x)] \\ &+ \left[-x^2\ln(x) + \frac{1}{2} + x - \frac{3}{2}x^2\right]\mathrm{S}_{1,2}(1-x) \\ &+ \frac{x^2}{4}\ln^3(x) - \frac{x}{4}\ln^2(x) - \frac{x}{2}\ln(x) - \frac{1}{2}x(1-x). \end{aligned} \tag{A.160}$$

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